Question

(a) If the Laplace transform of a suitable function $f(t)$ is defined to be

$$\bar{f}(p) = \int_0^\infty dt\ f(t)\exp(-pt),\ \ Re\ p > 0$$

state *carefully* the inverse transform in terms of a contour integral.

(b) **(i)** Using this integral representation, defining $-\pi < \arg\ p \leq \pi$, and suitably closing the contour, show that the inverse Laplace transform of $\bar{f}_1(p) = \log p$ is $f_1(t) = -\frac{1}{t}$. (You may assume without proof that all integrals over infinite or vanishing arcs give a zero contribution.)

(ii) Hence show that the inverse Laplace transform of $\bar{f}_2(p) = \log(p+1)$ is $f_2(t) = -\frac{e^{-t}}{t}$.

(iii) Using parts (i) and (ii), show that the inverse Laplace transform of $\bar{f}_3 = \log\left[\frac{(p+1)}{p}\right]$ is $f_3(t) = \frac{(1-e^{-t})}{t}$.

(c) Show that the following equation

$$t\frac{d^2 f}{dt^2} + 2\frac{df}{dt} + tf = 1 - 2e^{-t}$$

$$f(0) = 1,\ f'(0) = -\frac{1}{2}.$$

can be Laplace-transformed to give

$$\frac{d\bar{f}}{dp} = -\frac{1}{p(p+1)}.$$

Consequently, using the results of part b, solve for $f(t)$.

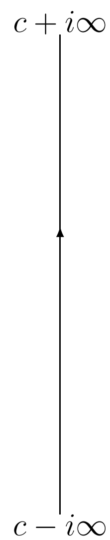
Answer

(a) $\hat{f}(p) = \int_0^\infty dt\ f(t)e^{-pt},\ Re(p) > 0$

$\Rightarrow f(t) = \frac{1}{2\pi i}\int_C dp\ e^{pt}\hat{f}(p),\ t > 0$

<u>Bromwich Contour</u>: C:

(p)



All singularities to left of c.

(b) **(i)** Want $f_1(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dp\ \ \log p e^{pt}$

p has branch point at $p = 0$, define branch on $-\pi < \arg\ p \leq \pi$.

Complete to <u>left</u>.

PICTURE

Close p to left and form keyhole contour.

Integrals (2), (4) and (6) vanish by hint.

Therefore

$$\begin{aligned}
(1) &= \frac{1}{2\pi i}\int_{\infty}^{0} dx\ e^{i\pi}\log(xe^{i\pi}e^{-xt} \\
& \quad (3): p = xe^{i\pi} \\
& \quad +\frac{1}{2\pi i}\int_{0}^{\infty} dx\ e^{i\pi}\log(xe^{-i\pi})e^{-xt} \\
& \quad (5): p = xe^{-i\pi} \\
&= +\frac{1}{2\pi i}\int_{0}^{\infty} dx(\log x + i\pi)e^{-xt} \\
& \quad -\frac{1}{2\pi i}\int_{0}^{\infty} dx(\log x - i\pi)e^{-xt} \\
&= \frac{2\pi i}{2\pi i}\int_{0}^{\infty} dx\ e^{-xt} \\
&= -\frac{1}{t}
\end{aligned}$$

Therefore $\underline{\hat{f}_1(p) = \log p \Leftrightarrow f_1(t) = -\frac{1}{t}}$.

(ii) Use Laplace's transform property that

$$L\left[e^{at}f(t)\right] = \hat{f}(p)$$

$\Rightarrow L\left[e^{-t}f(t)\right] = \hat{f}(p+1)$

$$\begin{aligned}
\text{Therefore if } \hat{\mathrm{f}}(\mathrm{p}+1) = \hat{\mathrm{f}}_2(\mathrm{p}) &= \log(p+1) \\
\Rightarrow \hat{f}_2(p) &= \hat{f}_1(p+1) \\
\Rightarrow f_2(t) &= e^{-t}f_1(t) \\
\Rightarrow \underline{f_2(t) = -\frac{e^{-t}}{t}} &
\end{aligned}$$

(iii) Laplace transforms are linear, so

$$L[f_1(t) + f_2(t)] = L[f_2(t)]$$

$$\begin{aligned}
\Leftrightarrow L\left[\frac{1-e^{-t}}{t}\right] &= L\left[\frac{1}{t}\right] + L\left[-\frac{e^{-t}}{t}\right] \\
&= -L\left[\frac{1}{t}\right] + L\left[-\frac{e^{-t}}{t}\right] \\
&= \log(p+1) - \log\ p \\
&= \log\left(\frac{p+1}{p}\right)
\end{aligned}$$

$$\Rightarrow \underline{L^{-1}\left[\log\left(\frac{p+1}{p}\right)\right] = \frac{1-e^{-t}}{t}}$$

(c)

$$\begin{aligned}
L[tf''] &= -\partial p L[f''] \\
&= -\frac{pl}{\partial p}[p^2\bar{f}(p) - p\underbrace{f(0_+)}_{=+1} + \underbrace{f'(0_+)}_{=-\frac{1}{2}}] \\
&= -\frac{\partial}{\partial p}\left[p^2\bar{f} - p - \frac{1}{2}\right] \\
&= -2p\bar{f} - p^2\bar{f}' + 1 \\
L[2f'] &= 2[p\bar{f} - \underbrace{f(0_+)}_{=+1}] \\
&= 2p\bar{f} - 2 \\
L[tf] &= -\partial_p\bar{f}(p) = -\bar{f}'(p) \\
L[1] &= \int_0^\infty dt\ e^{-pt} = \frac{1}{p} \\
L[-2e^{-t}] &= -2\int_0^\infty dt\ e^{-(p+1)t} = \frac{-2}{(p+1)}
\end{aligned}$$

Therefore L[equation] is

$$-2p\bar{f} - p^2\bar{f}' + 1 + 2p\bar{f} - 2 - f' = \frac{1}{p} - \frac{2}{p+1}$$

Therefore

$$\begin{aligned}
-(p^2+1)\bar{f}' &= 1+\frac{1}{p}-\frac{2}{p+1} \\
&= \frac{p(p+1)+(p+1)-2p}{p(p+1)} \\
&= \frac{p^2+p+p+1-2p}{p(p+1)} \\
\text{Therefore } -(\mathrm{p}^2+1)\bar{\mathrm{f}}' &= \frac{(p^2+1)}{p(p+1)} \\
\text{Therefore } \bar{\mathrm{f}}' &= \underline{-\frac{1}{p(p+1)}} \text{ as required}
\end{aligned}$$

$$\begin{aligned}
\bar{f}' &= -\frac{1}{p}+\frac{1}{p+1} \\
\Rightarrow \bar{f} &= -\log\ p+\log(p+1)+\log\ c \quad c = const \\
\bar{f} &= \log\left\{\left[\frac{(p+1)}{p}\right]c\right\} \\
\Rightarrow f(t) &= \underline{D\frac{(1-e^{-t})}{t}}
\end{aligned}$$

By previous bits of question and standard results. $D = const$
But if $f(0) = 1 \Rightarrow D = 1$ since

$$f(t) = \left(\frac{1-e^{-t}}{t}\right)$$

and $\lim_{t\to 0} f(t) = \lim_{t\to 0} +\frac{e^{-t}}{1} = 1$ by L'Hopital
Hence

$$\underline{f(t) = \frac{1-e^{-t}}{t}}$$