Question

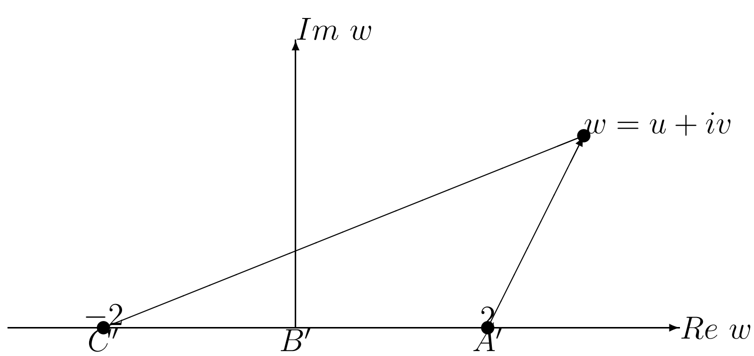
(a) Define carefully what is meant by a *conformal map*, $w = f(z)$

(b) Let $z = x + iy,\ w = u + iv$ and consider the Joukowski transformation

$$w = z + \frac{1}{z}.$$

Show that this transformation maps the region $Im(z) > 0,\ |z| > 1$ to the region $Im\, 0$ (i.e., the shaded portions on the diagram below).

PICTURE



(c) By considering the imaginary part of the complex function

$$\alpha \log(w+2) + \beta \log(w-2) + \gamma,$$

where $\alpha,\ \beta,\ \gamma$ are real constants to be found, write down a harmonic function ϕ which satisfies the boundary conditions

$$\phi(u, 0^+) = \begin{cases} 0, |u| > 2 \\ 1, |u| < 2 \end{cases}$$

(Hint: take $-\pi < \arg(w+2) \leq \pi$ and $-\pi < \arg(w-2) \leq \pi$.)

(d) Hence solve the equation $\bigtriangledown^2 F(x,y) = 0$ in the region $y \geq 0,\ |x^2+y^2| \geq 1$, subject to the boundary conditions, leaving your answer in terms of $z = x + iy$

$$F(x,y) = \begin{cases} 0, & y=0,\ |x| > 1 \\ 1, & y=0,\ |x| < 1 \end{cases}$$

Answer

(a) A conformal map f on ???? is one which preserves angles (and also the sense of the angle). A differentiable function gives conformal transformations, provided $f'(z) \neq 0$.

(b) Consider Joukowski:

$$w = f(z) = z + \frac{1}{z}$$

Take $|z| = 1,\ Im(z) > 0$ with $z = e^{i\theta},\ 0 < \theta < \pi$

$w = e^{i\theta} + e^{-i\theta} = 2\cos\theta;\ 0 < \theta < \pi$

so $ABC \longrightarrow A'B'C'$

since $-2 < 2\cos\theta < 2$

Take $Im(z) = 0,\ Re(z) = x < -1$

Therefore $w = x + \dfrac{1}{x}$ with runs between $w = -1 - \dfrac{1}{1} = -2$ and $w = -\infty + \dfrac{1}{-\infty} = -\infty$

so $-\infty cz \rightarrow -\infty c'$

Likewise for $Im(z) = 0,\ Re(z) = x > 1$

$$A\infty \to A'\infty$$

Pick point in $Im(z) > 0,\ |z| > 1$ and see where it goes, e.g.,
$z = 2i \Rightarrow = 2i + \dfrac{1}{2i} = \left(2 - \dfrac{1}{2}\right) i = \dfrac{3}{2} i$
which has $Im(w) > 0$.

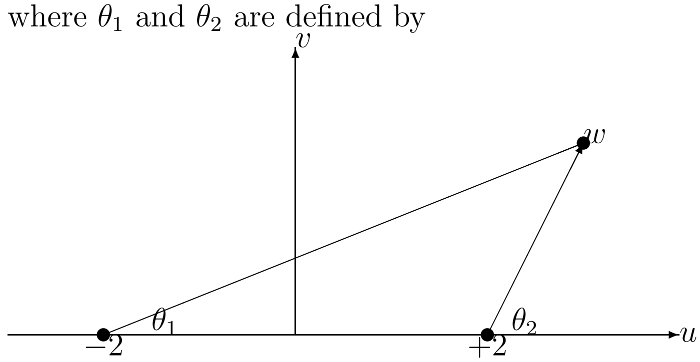
Thus transformation is as stated in question.

(c) Imaginary part of

$$\alpha \log(w + 2) + \beta \log(w - 2) + \gamma,\ \alpha,\ \beta\ \gamma \text{ real}$$

is $\alpha\theta_1 + \beta\theta_2 + \gamma$

where θ_1 and θ_2 are defined by



$$\Phi(w) = \alpha \log(w + 2) + \beta \log(w - 2) + \gamma$$

is analytic, except at $w = \pm 2$.

Thus $Im(\Phi(w))$ must be harmonic, except at those points and hence satisfies Laplace's equation in (u, v).

To satisfy

$$\phi(u, 0^+) = \left\{ \begin{array}{ll} 0, & |u| > 2 \\ 1, & |v| < 2 \end{array} \right\}$$

we have on

(A) $Re(u) > 0,\ |u| > 2;\ \theta_1 = 0,\ \theta_2 = 0$

$$\text{Therefore } 0 = \alpha \cdot 0 + \beta \cdot 0 + \gamma \Rightarrow \underline{\gamma = 0}$$

(B) $Re(u) > 0,\ |u| < 2;\ \theta_1 = 0,\ \theta_2 = \pi$

$$\text{Therefore } 1 = \alpha \cdot 0 + \beta \cdot \pi + 0 \Rightarrow \underline{\beta = \frac{1}{\pi}}$$

(C) $Re(u) > 0,\ |u| < 2;\ \theta_1 = 0,\ \theta_2 = 0$

$$\text{So same as above} \underline{\beta = \frac{1}{\pi}}$$

(D) $Re(u) > 0,\ |u| > 2;\ \theta_1 = \pi,\ \theta_2 = \pi$

$$\text{Therefore } 0 = \alpha \cdot \pi + \beta \cdot \pi \Rightarrow \underline{\alpha = -\frac{1}{\pi}}$$

Therefore

$$\underline{\Phi(w) = \frac{1}{\pi}\log(w-2) - \frac{1}{\pi}\log(w+2)}$$

$$\underline{\phi = \frac{\theta_2}{\pi} - \frac{\theta_1}{\pi} = \frac{1}{\pi}\arctan\left(\frac{v}{u-2}\right) - \frac{1}{\pi}\arctan\left(\frac{v}{u+2}\right)}$$

(d) Avoiding $z = 0$ we have from theorem in notes that image of harmonic ϕ in $w = f(z)$ is also harmonic.

So given that Joukowski transform

$$w = z + \frac{1}{z}$$

we have the boundary conditions of $F(x, y)$ mapping onto the boundary conditions of $\phi(x, y)$.

Hence we have that

$Im(\Phi(w(z)))$ satisfies $\nabla^2 F(x, y) = 0$ in given region of z with boundary conditions.

Therefore

$$Im[\Phi(w(z))] = Im\left[\frac{1}{\pi}\log\left(z + \frac{1}{z} - 2\right) - \frac{1}{i}\log\left(z + \frac{1}{z} + 2\right)\right]$$

$$= Im\left[\frac{1}{\pi}\log\left(x + iy + \frac{1}{x+iy} - 2\right) - \frac{1}{\pi}\left(x + iy + \frac{1}{x+iy} + 2\right)\right]$$