Question

The wave equation with constant speed $c > 0$ is given by

$$c^2 u_{xx} - u_{tt} = 0.$$

(a) Classify this equation and identify whether it is elliptic, parabolic or hyperbolic in the (x, y) plane. Hence *state* the standard from of the equation in characteristic coordinates ξ, η and show that the general solution is

$$u(x,t) = F(x - ct) + G(x + ct)$$

for arbitrary functions F and G.

(b) Show that, in general, the following initial value system

$$\begin{aligned} c^2 u_{xx} - u_{tt} &= 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

$$0 < x < \pi, t > 0$$

may be solved to give

$$u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} ds g(s).$$

You must show clear working and carefully define any parameters you use.

(c) Write down the solution of the wave equation in the following specific cases

$$\begin{array}{lll} i) & f(x) = 0 & g(x) = \sin x, \\ ii) & f(x) = \dfrac{1}{(x+1)} & g(x) = 0, \\ iii) & f(x) = 0 & g(x) = 1, \\ iv) & f(x) = 0 & g(x) = \dfrac{1}{(x^2+1)}. \end{array}$$

Identify any singularities of $u(x, t)$ in the region $t > 0$ and hence comment on the validity of each solution.

Answer

$$c^2 u_{xx} - u_{tt} = 0$$

2nd order linear homogeneous constant coefficient PDE.
Discriminant: $a = c^2,\ b = 0,\ c = -1$

$$b^2 - a\text{“}c\text{”} = +c^2 > 0 \ \text{ everywhere}$$

Therefore hyperbolic equation everywhere
Characteristic coordinates given by:
$cdt^2 - dx^2 = 0$
$\Rightarrow (cdt - dx)(cdt + dx) = 0$
$\Rightarrow cdt - dx = 0$ or $cdt + dx = 0$
$\Rightarrow \dfrac{dx}{dt} = +c \quad \dfrac{dx}{dt} = -c$
$\Rightarrow x = ct + \xi \quad x = ct + \eta$
Therefore $\xi = (x - ct);\ \eta = (x + ct)$
and hyperbolic wave equation transforms to

$$u_{\xi\eta} = 0$$

$$\Rightarrow u = F(\xi) + G(\eta)$$

Hence

$$u(x,t) = F(x - ct) + G(x + ct) \quad F,\ G \text{ arbitrary}$$

(i) $f(x) = 0,\ g(x) = \sin x$

$$\begin{aligned}
\Rightarrow u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s\, ds \\
&= \frac{1}{2c}[-\cos s]_{x-ct}^{x+ct} \\
&= \frac{1}{2c}[-\cos(x+ct) + \cos(x-ct)] \\
&= \frac{1}{2c}[-\cos x\cos ct + \sin x \sin ct \\
&\quad + \cos x \cos ct + \sin x \sin ct] \\
\text{Therefore u(x, t)} &= \underline{+\frac{1}{c}\sin x \sin ct}
\end{aligned}$$

Solution valid for all t, for all finite x.

(ii) $f(x) = \dfrac{1}{x+1}, \ g(x) = 0$

$$\begin{aligned} \Rightarrow u(x,t) &= \frac{1}{2}\left[\frac{1}{(x+ct+1)} + \frac{1}{(x-ct+1)}\right] \\ &= \frac{1}{2}\frac{2x+2}{(1+x)^2 - c^2t^2} \\ &= \underline{\frac{x+1}{(1-x)^2 - c^2t^2}} \end{aligned}$$

Solution has a singularity when $1 + x = \pm ct$ so not a sensible result.

(iii) $f(x) = 0, \ g(x) = 1$

Therefore

$$\begin{aligned} u(x,t) &= \frac{1}{2}(0+0) + \frac{1}{2c}\int_{x-ct}^{x+ct} ds \\ &= \frac{1}{2c}(x + ct - x + ct) \\ &= \underline{t} \end{aligned}$$

Solution grows with t. Not sensible as $t \to \infty$.

(iv) $f(x) = 0, \ g(x) = \dfrac{1}{1+x^2}$

$$\begin{aligned} u(x,t) &= \frac{1}{2}(0+0) + \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{ds}{1+s^2} \\ &= \frac{1}{2c}[\arctan s]_{x-ct}^{x+ct} \\ &= \frac{1}{2c}[\arctan(x+ct) - \arctan(x-ct)] \end{aligned}$$

Assuming arctan defined on specified range, solution is valid flor all x, for all t (as $t \to \infty$) and x finite.

$$u(x,t) \to \frac{1}{2c}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{\pi}{2c}$$

(NB energy considerations if x larger $\cdots$)