Question
The following notation is used in this question.

$$y = y(x),\ y' = \frac{dy}{dx},\ y'' = \frac{d^2y}{dx^2}.$$

You may assume that al y are sufficiently differentiable functions.

(a) For what A does the following functional have an extremal? Find the associated value of I.

$$I = \int_0^1 dx\ (\frac{1}{2}y^2 + 2y'y + ye^x)\ ,$$

$$y(0) = -1,\ y'(0) = A$$

(b) Find the extremal value of J where,

$$J = \int_0^1 dx\ (y + 2xy' + \frac{1}{2}y''^2)$$

$$y(0) = 0,\ y'(0) = 0,\ y(1) = \frac{1}{24},\ y'(1) = \frac{1}{6}.$$

Be sure that you carefully explain your methods.

Answer

(a)

$$I = \int_0^1 dx\ \left(\frac{y^2}{2} + 2y'y + ye^x\right)$$

$$y(0) = 1,\ y'(0) = A$$

Euler-Lagrange:

$\dfrac{dF}{dy} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ with

$$F = \frac{y^2}{2} + 2y'y + ye^x$$

$$\frac{\partial F}{\partial y} = y + 2y' + e^x$$

$$\frac{\partial F}{\partial y'} = 2y$$

Therefore $\quad y + 2y' + e^x - \dfrac{d}{dx}(2y) = 0$

Therefore $\quad y + e^x = 0$

Therefore $\quad \underline{y = -e^x}$

Therefore

$$\begin{aligned} y(0) &= -e^{-0} = -1 \quad \surd\surd \\ y'(0) &= -e^x|_{x=0} = -1 \end{aligned}$$

Therefore $\underline{\text{need } A = -1}$.

Therefore with $y = -e^x$

$$\begin{aligned} I &= \int_0^1 dx \left(\frac{e^{2x}}{2} + 2e^{2x} - e^{2x} \right) \\ &= \frac{3}{2} \int_0^1 dx\ e^{2x} \\ &= \frac{3}{2} \left[\frac{e^{2x}}{2} \right]_0^1 \\ I &= \underline{\frac{3}{4}(e^2 - 1)} \end{aligned}$$

(b) $I = \displaystyle\int_0^1 dx \left(y + 2xy' + \frac{y''^2}{2} \right)$

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \\ y(1) &= \frac{1}{24} \\ y'(1) &= \frac{1}{6} \end{aligned}$$

$$F = y + 2xy' + \frac{y''^2}{2}$$

Euler-Lagrange equation is:

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y} = 0$$

$$\frac{\partial F}{\partial y''} = \frac{2y''}{2};\ \frac{\partial F}{\partial y'} = 2x;\ \frac{\partial F}{\partial y} = 1$$

Therefore $\frac{d^2}{dx^2}\left(\frac{2y''}{2}\right) - \frac{d}{dx}(2x) + 1 = 0$

Therefore $\frac{2y''''}{2} - 2 + 1 = 0$

Therefore $\underline{y'''' = 1}$

Therefore

$$\begin{aligned} y''' &= x + A \\ y'' &= \frac{x^2}{2} + Ax + B \\ y' &= \frac{x^3}{6} + \frac{Ax^2}{2} + Bx + C \\ y &= \frac{x^4}{24} + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \end{aligned}$$

Boundary conditions:

$$\begin{aligned} y(0) = 0 &\quad\Rightarrow\quad D = 0 \\ y'(0) = 0 &\quad\Rightarrow\quad C = 0 \\ y(1) = \frac{1}{24} &\quad\Rightarrow\quad \frac{1}{24} = \frac{1}{24} + \frac{A}{6} + \frac{B}{2} \\ &\quad\Rightarrow\quad \underline{A + 3B = 0} \quad (1) \end{aligned}$$

$$\begin{aligned} y'(1) = \frac{1}{6} &\quad\Rightarrow\quad \frac{1}{6} = \frac{1}{6} + \frac{A}{2} + B \\ &\quad\Rightarrow\quad \underline{A + 2B = 0} \quad (2) \end{aligned}$$

$(1) - (2) \ \Rightarrow B = 0 \Rightarrow A = 0$

Therefore $y = \dfrac{x^4}{24}$

Hence

$$
\begin{aligned}
I &= \int_0^1 dx \ \left(\frac{x^4}{24} + 2x \cdot \frac{x^3}{6} + \frac{1}{2}\left(\frac{x^2}{2}\right)^2 \right) \\
&= \int_0^1 dx \ x^4 \left(\frac{1}{24} + \frac{1}{3} + \frac{1}{8}\right) \\
&= \frac{1}{5} \times \left(\frac{1}{2}\right) \\
&= \underline{\frac{1}{10}}
\end{aligned}
$$

$$u = F = G$$

$u(x,0) = F(x) + G(x0 = f(x)$

$u_t(x,0) = -cF'(x) + cG'(x) = g(x)$

$F + G = f$

$-F' + G' = \dfrac{1}{c}g$

$-F(x) + G(x) = \dfrac{1}{c}\displaystyle\int_A^x g(s)\, ds$

$F + G = f$

$G(x) = \dfrac{1}{2}f(x) + \dfrac{1}{2c}\displaystyle\int_A^x g(s)\, ds$

$F(x) = \dfrac{1}{2}f(x) - \dfrac{1}{2c}\displaystyle\int_A^x g(s)\, ds$

$F(x-ct) + G(x+ct) = \dfrac{1}{2}z[f(x-ct) + f(x+ct)]$

$?????(x,t) = \dfrac{1}{2}\left[f(x-ct) + f(x+ct) + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\, ds\right]$

$+\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s) ds + \dfrac{1}{2c}\int_{x-???ct}^{A} g(s)\, ds$