Question

This question will demonstrate that a convergent Taylor series can also be an asymptotic series. Consider the Taylor series expansion about the origin of a sufficiently differentiable function $f(x)$:

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \cdots + \frac{1}{n!}x^n f^{(n)}(0) + R_n(x)$$

where $R_b(x)$ has the exact representation

$$R_n(x) = \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\, dt.$$

(i) Write down the Poincaré definition of an expansion with a gauge $\{x^n\}$ which is asymptotic to a function $f(x)$ at $x = 0$

(ii) Using the fact that $\left|\int_0^z g(\zeta)\, d\zeta\right| \leq \int_0^z |g\zeta| d\zeta$ for any suitable integrable integrand, show that

$$|R_n(x)| \leq \frac{|x^n|}{n!}\int_0^x \left(1 - \frac{t}{x}\right)^n \left|f^{(n+1)}(t)\right|\, dt.$$

(iii) Assuming that $f^{(n+1)}(t)$ is continuous on $[0, x]$ then there exists a number M, such that $|f^{(n+1)}(x)| \leq M$. Given this, show that,

$$|R_n(x)| \leq \frac{M|x|^{n+1}}{n!}\int_0^1 (1-\xi)^n\, d\xi$$

(iv) Hence show that

$$R_n(x) = O(x^{n+1}),$$

and establish that the Taylor expansion is indeed Poincaré asymptotic to $f(x)$ at the origin.

Answer

(i) $f(x) \sim \sum_{s=0}^{\infty} a_s x^{+s}$ as $x \to 0^+$ if $\left[f(x) - \sum_{s=0}^{n} a_s x^s\right] = 0(x^{n+1})$ as $x \to 0^+$ $\quad (A)$

for every *fixed* $n \geq 0$.

With gauge $\{x^n\}$ as $x \to 0$. This is the usual Taylor series expansion which will have a finite radius of convergence.

(ii) Given

$$R_n(x) = \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)})(t)\,dt$$
$$|R_n(x)| = \frac{1}{n!}\left|\int_0^x (x-t)^n f^{(n+1)})(t)\,dt\right|$$

we have

$$\begin{aligned} |R_n(x)| &\leq \frac{1}{n!}\int_0^x |(x-t)^n f^{(n+1)}(t)|\,dt \\ &= \frac{|x^n|}{n!}\int_0^x \left|\left(1-\frac{t}{x}\right) f^{(n+1)}(t)\right|\,dt \end{aligned}$$

But consider the range of the integrand: t runs from $0 \to x$

i.e., $0 \leq t \leq x$. Therefore $\left|\left(1-\frac{}{x}\right)^n\right| = \left(1-\frac{t}{x}\right)^n$ for every positive integer n.

Therefore $|R_n(x)| \leq \dfrac{|x^n|}{n!}\displaystyle\int_0^x \left(1-\frac{t}{x}\right)^n \left|f^{(n+1)}(t)\right|\,dt$ as required.

(iii) Given $f^{(n+1)}(t) \in C[0,x] \Rightarrow$ there exists $M(\geq 0)$ such that

$$\left|f^{(n+1)}(t)\right| \leq M.$$

Thus in (ii),

$$|R_n(x)| \leq M\frac{|x^n|}{n!}\int_0^x \left(1-\frac{t}{x}\right)^n\,dt$$
$$\Rightarrow |R_n(x)| \leq \frac{M|x^{n+1}|}{n!}\int_0^1 (1-\xi)^n\,d\xi$$

(setting $\frac{t}{x} = \xi$)

(iv) $\displaystyle\int_0^1 (1-\xi)^n \, d\xi \overset{(\xi=\cos^2 u)}{=} 2\int_0^{\frac{\pi}{2}} \sin^{n+1} u \cos u \, du = 2\left[\frac{\sin^{n+2} u}{n+2}\right]_0^{\frac{\pi}{2}} = 2$

Therefore $|R_n(x)| \leq \dfrac{2M}{(n+2)n!}\left|x^{n+1}\right|$

So, by definition of order symbols,

$R_n(x) = O(x^{n+1})$ $\left[\text{implied constant } = \dfrac{2M}{(n+2)n!}\right]$

Clearly this satisfies (A) as $x \to 0^+$ so Taylor series is *also* asymptotic.