Question

Find the general solution of the following non-homogeneous equations:

1. $y'' - 5y' + 6y = 2e^x$
2. $y'' - 2y' - 3y = 3e^{2x}$ $\qquad (*)$
3. $y'' + 2y' + 5y = 3\sin 2x$
4. $y'' + 2y' + y = 3e^{-x}$
5. $y'' + \lambda^2 y = \cos \omega x, \qquad (*)$
   Here ω and λ are positive constants.
   i) Solve for the case $\quad \omega^2 \neq \lambda^2$
   ii) Solve for the case $\quad \omega = \lambda$

Answer

1. Auxiliary equation is: $m^2 - 5m + 6 = 0 \qquad$ giving $m = 2, 3$
   Hence the complementary function is: $y_{cf} = Ae^{2x} + Be^{3x}$
   Try the particular integral: $y_p = ae^x$
   Equation becomes: $ae^x - 5ae^x + 6ae^x = 2e^x$
   so that $2ae^x = 2e^x$
   and hence $\qquad a = 1$
   The general solution is: $y = Ae^{2x} + Be^{3x} + e^x$
2. Auxiliary equation is: $m^2 - 2m - 3 = 0 \qquad$ giving $m = 3, -1$
   Hence the complementary function is: $y_{cf} = Ae^{3x} + Be^{-x}$
   Try the particular integral: $y_p = ae^{2x}$
   Equation becomes: $(4a - 4a - 3a)e^{2x} = 3e^{2x}$ so that $a = -1$
   The general solution is: $y = Ae^{3x} + Be^{-x} - e^{2x}$
3. Auxiliary equation is: $m^2 + 2m + 5 = 0 \qquad$ giving $m = -1 + 2i, -1 - 2i$
   Hence the complementary function is: $y_{cf} = e^{-x}(D\sin(2x) + E\cos(2x))$
   Try the particular integral: $y_p = a\sin(2x) + b\cos(2x)$

Equation becomes:
$-4a\sin(2x) - 4b\cos(2x) + 2(2a\cos(2x) - 2b\sin(2x)) + 5(a\sin(2x) + b\cos(2x)) = 3\sin(2x)$
so that

$$\begin{aligned}(-4a - 4b + 5a)\sin(2x) &= 3\sin(2x)\\ (-4b + 4a + 5b)\cos(2x) &= 0\end{aligned}$$

which implies $a - 4b = 3 \qquad 4a + b = 0$
So that $a = 3/17$ and $b = -12/17$
The general solution is:

$$y = e^{-x}(D\sin(2x) + E\cos(2x)) + \frac{3}{17}\sin(2x) - \frac{12}{17}\cos(2x)$$

4. Auxiliary equation is: $m^2 + 2m + 1 = 0 \qquad$ giving $m = -1, -1$
Hence the complementary function is: $y_{cf} = e^{-x}(A + Bx)$
Try the particular integral: $y_p = ax^2e^{-x}$
(note you can try $y_p = bxe^{-x}$ but will find this does not work. This is because the complementary function contains both e^{-x} and xe^{-x})
Equation becomes:
$\left(2ae^{-x} - 4axe^{-x} + ax^2e^{-x}\right) + 2\left(2axe^{-x} - ax^2e^{-x}\right) + \left(ax^2e^{-x}\right) = 3e^{-x}$
and simplifying gives: $2ae^{-x} = 3e^{-x}$ so that $a = 3/2$
The general solution is: $y = e^{-x}(A + Bx) + \frac{3}{2}x^2e^{-x}$

5. (i) Auxiliary equation is: $m^2 + \lambda^2 = 0 \qquad$ giving $m = \pm\lambda i$
Hence the complementary function is: $y_{cf} = A\cos(\lambda x) + B\sin(\lambda x)$
Try the particular integral: $y_p = a\cos(\omega x) + b\sin(\omega x)$
Equation becomes:
$-\omega^2(a\cos(\omega x) + b\sin(\omega x)) + \lambda^2(a\cos(\omega x) + b\sin(\omega x)) = \cos(\omega x)$
so that

$$\begin{aligned}(-\omega^2 b + \lambda^2 b)\sin(\omega x) &= 0\\ (-\omega^2 a + \lambda^2 a)\cos(\omega x) &= \cos(\omega x)\end{aligned}$$

which implies that $b = 0$ and $a = \dfrac{-1}{-\omega^2 + \lambda^2}$
(note that this requires that $\omega \neq \lambda$ or a will not exist)
The general solution is:

$$y = A\cos(\lambda x) + B\sin(\lambda x) + \frac{1}{\omega^2 - \lambda^2}\cos(\omega x)$$

(ii) This is exactly as for the previous part of the question but we must find a different particular integral.
The complementary function is: $y_{cf} = A\cos(\lambda x) + B\sin(\lambda x)$
Note that the right hand side of the equation contains functions in the complementary function hence try the particular integral:
$y_p = ax\cos(\lambda x) + bx\sin(\lambda x)$
Equation becomes:
$a(-\lambda^2 x\cos(\lambda x) - 2\lambda\sin(\lambda x)) + b(-\lambda^2 x\sin(\lambda x) + 2\lambda\cos(\lambda x))) + \lambda^2(ax\cos(\lambda x) + bx\sin(\lambda x)) = \cos(\lambda x)$
so that

$$-2a\lambda\sin(\lambda x) + 2b\lambda\cos(\lambda x) = \cos(\lambda x)$$

which implies that $a = 0$ and $b = \dfrac{1}{2\lambda}$
The general solution is:

$$y = A\cos(\lambda x) + B\sin(\lambda x) + \frac{1}{2\lambda}x\sin(\omega x)$$