Question

(i) $\int x \sin x \, dx$

(ii) $\int_0^1 x^2 \exp(x) \, dx$

(iii) $\int \arctan x \, dx$

(iv) $\int \exp(2x) \cos(3x) \, dx$

Answer

(i)

$\int x \sin x \, dx$

Use integration by parts

$$\begin{array}{cc} u = x & \frac{dv}{dx} = \sin x \\ \frac{du}{dx} = 1 & v = -\cos x \end{array}$$

NB forget arbitrary constants at this stage

Now

$$\int \left(u \frac{dv}{dx} \right) dx = uv - \int \left(v \frac{du}{dx} \right) dx + const$$

$$\begin{aligned} \Rightarrow \int x \sin x \, dx &= -x \cos x + \int \cos x \, dx + c \\ &= \underline{-x \cos x + \sin x + c} \end{aligned}$$

Check by differentiation.

(ii)

$\int_0^1 x^2 \exp(x) \, dx$

Use integration by parts

$$\begin{array}{cc} u = x^2 & \frac{dv}{dx} = e^x \\ \frac{du}{dx} = 2x & v = e^x \end{array}$$

Therefore $$\left. \begin{aligned} \int_0^1 x^2 e^x \, dx &= [uv]_0^1 - \int_0^1 \left(v \frac{du}{dx} \right) dx \\ &= [x^2 e^x]_0^1 - 2 \int_0^1 x e^x \, dx \end{aligned} \right\} \quad A$$

The second integral can also be evaluated by parts again:

Consider $\int_0^1 2xe^x\,dx$

New $u,\ v$: $\left.\begin{array}{ll} u = 2x & \dfrac{dv}{dx} = e^x \\ \dfrac{du}{dx} = 2 & v = e^x \end{array}\right\}\quad B$

So $\int_0^1 2xe^x\,dx = [2xe^x]_0^1 - \int_0^1 2e^x\,dx$ Can be checked easily.

Collect A and B together

$$\begin{aligned}
\int_0^1 x^2e^{2x}\,dx &= [x^2e^x]_0^1 - \left\{[2xe^x]_0^1 - 2\int_0^1 e^x\,dx\right\} \\
&= [x^2e^x - 2xe^x]_0^1 + 2\int_0^1 e^x\,dx \\
&= (1\cdot e^1 - 2\cdot 1\cdot e^1) - (0-0) + 2[e^x]_0^1 \\
&= -e + 2[e^1 - e^0] \\
&= -e + 2(e-1) \\
&= \underline{e-2}
\end{aligned}$$

(iii)

$\int \arctan x\,dx$

Integrate by parts: "$\arctan x = 1 \times \arctan x$"

$$\begin{array}{ll} u = \arctan x & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = \dfrac{1}{1+x^2} & v = x \end{array}$$

(standard result)

Therefore

$$\begin{aligned}
\int \arctan x &= uv - \int \frac{du}{dx}\,dx \\
&= (x\arctan x) - \int \frac{x}{1+x^2}\,dx \\
&\quad \frac{f'}{f}\ \text{integral : can spot answer or do by x} \times \text{v} \\
&= \underline{x\arctan x - \frac{1}{2}\log(1+x^2) = c}
\end{aligned}$$

(iv)

$I = \int \exp(2x)\cos(3x)\,dx$

Integrate by parts

$$\begin{array}{ll} u = cos3x & \dfrac{dv}{dx} = e^{2x} \\ \dfrac{du}{dx} = -3\sin 3x & v = \dfrac{e^{2x}}{2} \end{array}$$

$$A: \quad I = \int e^{2x}\cos 3x\,dx = \frac{e^{2x}}{2}\cos 3x - \int \left(-\frac{3}{2}\sin 3x e^{2x}\right)\,dx$$
$$= \frac{e^{2x}}{2}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx$$

Still difficult to do, but repeat integration by parts:

$\int e^{2x}\sin 3x\,dx$

New $u,\ v$:

$$u = \sin 3x \qquad \frac{dv}{dx} = e^{2x}$$
$$\frac{du}{dx} = 3\cos 3x \qquad v = \frac{e^{2x}}{2}$$

$$B: \int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{2}\sin 3x - \int \frac{3}{2}e^{2x}\cos 3x\,dx$$

Original integral back again! oh dear? No good!

Combine A and B:

$$I = \frac{e^{2x}}{2}\cos 3x + \frac{3}{2}\quad B$$
$$= \frac{e^{2x}}{2}\cos 3x + \frac{e^{2x}}{2}\cos 3x + \frac{3}{2}\left[\frac{e^{2x}\sin 3x}{2} - \frac{3}{2}I\right]$$
$$\Rightarrow \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}I$$

This is an equation in the unknown I!

Solve for I:

$$\frac{13}{4}I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x$$
$$\Rightarrow \int e^{2x}\cos 3x\,dx = \underline{\frac{2}{13}e^{2x}\left(\cos 3x + \frac{3}{2}\sin 3x\right) + c}$$

NB Don't forget arbitrary constant at end.

Check by differentiation.