Question

(i) $\int x(3x^2+7)\,dx$

(ii) $\int \sin^4 x \cos x\,dx$

(iii) $\int \cos^5 x\,dx$

(iv) $\int \cos^4 x\,dx$

(v) $\int \sec x\,dx$

(vi) $\int \frac{x\,dx}{\sqrt{x-2}}$

(vii) $\int \sin^2 x \cos^3 x\,dx$

(viii) $\int_1^2 \frac{8x\,dx}{(2x+1)^3}$

(ix) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cot x}{\sqrt{\csc^3 x}}\,dx$

(x) $\int_0^3 \frac{3\,dx}{\sqrt{9-x^2}}$

(xi) $\int \frac{(x-2)\,dx}{(x^2-4x-5)}$

(xii) $\int \frac{dx}{(x^2+6x+17)}$

(xiii) $\int \frac{(2x+5)\,dx}{(x^2+4x+5)}$

(xiv) $\int \frac{x^2\,dx}{(x+1)}$

Answer

(i)

$\int x(3x^2+7)\,dx$

either $\int(3x^2+7x)\,dx = \underline{\frac{3x^4}{4}+\frac{7x^2}{2}+c}\quad A$

or spot that $\frac{d}{dx}[(3x^2+7)^2+\bar{c}] = 2\cdot 6x(3x^2+7) = 12x(3x^2+7)$

so that "by inspection",

$x(3x^2+7) = \frac{1}{12}\frac{d}{dx}(3x^2+7)^2$

or $\int x(3x^2+7) = \frac{1}{12}(3x^2+7)^2 + c_1\quad B$

Check: evaluate B:

$\frac{1}{12}(3x^2+7)^2+c_1 = \frac{1}{12}(9x^4+42x^2+49)+c_1 = \frac{3}{4}x^4+\frac{7x^2}{2}+\left(\frac{49}{12}+c_1\right)$

Since it's an indefinite integral and integration constants are arbitrary $A = B$ if $c = \frac{49}{12}+c_1$.

Of course with integrands as simple as this you'd do it using method A. However, if I had asked for $\int x(3x^2+7)^4\,dx$, method B might have been quicker on spotting that $\int x(3x^2+7)^4\,dx = \frac{d}{dx}\left[\frac{1}{30}(3x^2+7)^5+c\right]$ ....

(ii)

$\int \sin^4 x\cos x\,dx$: spot that

$\frac{d}{dx}(\sin^5 x) = 5\sin^4 x\cos x \Rightarrow \int \sin^4 x\cos x\,dx = \frac{1}{5}\int\frac{d}{dx}(\sin^5 x)\,dx = \underline{\frac{1}{5}\sin^5 x + c}$

(check by differentiation)

NB $\int \frac{d(f(x))}{dx}\,dx = f(x)+c$ Standard result: the integral undoes the derivative.

(iii)

$$\begin{aligned}
\int\cos^5 x\,dx &= \int\cos^4 x\cos x\,dx\\
&= \int(1-\sin^2 x)^2\cos x\,dx\\
&= (1-2\sin^2 x+\sin^4 x)\cos x\,dx\\
&= \int(\cos x - 2\sin^2 x\cos x + \sin^4 x\cos x)\,dx\\
&= \underline{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c}
\end{aligned}$$

NB $\cos^2 x = 1-\sin^2 x$ and $\int\sin^2 x\cos x = \frac{1}{3}\sin^3 x + c$

(iv)

$$\begin{aligned}
\int \cos^4 x\,dx &= \int (\cos^2 x)^2\,dx \\
&= \frac{1}{4}\int (1+\cos 2x)^2\,dx \\
&\quad \text{(since } 2\cos^2 x - 1 = \cos 2x\text{)} \\
&= \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\,dx \\
&= \frac{1}{4}\int \left[1+2\cos 2x+\frac{1}{2}(1+\cos 4x)\right]\,dx \\
&\quad \text{(since } 2\cos^2(2x) - 1 = \cos 4x\text{)} \\
&= \frac{1}{4}\int \left(\frac{3}{2}+2\cos 2x+\frac{1}{2}\cos 4x\right)\,dx \\
&= \frac{1}{4}\left(\frac{3}{2}x+\sin 2x+\frac{1}{8}\sin 4x\right)+c \\
&= \underline{\frac{3x}{8}+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c}
\end{aligned}$$

(v)

$$\int \sec x\,dx$$

This can be written as a function of the type $\dfrac{f'(x)}{f(x)}$

How? Consider

$f(x) = \sec x + \tan x$
$f'(x) = \sec x \tan x + \sec^2 x$ (standard derivative)

Thus $\dfrac{f'(x)}{f(x)} = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \dfrac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} = \underline{\sec x}$

Now:

$\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln[f(x)] + c$ (standard integral)

So

$$\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)}\,dx = \int \frac{f'(x)}{f(x)}\,dx$$

where $f(x) = \sec x + \tan x$

Therefore $\underline{\displaystyle\int \sec x\,dx = \ln[\sec x + \tan x] + c}$

(vi)

$$\int \frac{x\,dx}{\sqrt{x-2}}$$

Try substitution

$$u = \sqrt{x-2} = (x-2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2}(x-2)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dx}{du} = 2(x-2)^{\frac{1}{2}}$$

and also $u^2 = x - 2 \Rightarrow x = u^2 + 2$

So

$$\begin{aligned} \int \frac{x\,dx}{\sqrt{x-2}} &= \int \frac{x}{\sqrt{x-2}} \frac{dx}{du} du \\ &= \int \frac{(u^2+2)}{\sqrt{x-2}} 2\sqrt{x-2} du \\ &= 2\int (u^2+2)\,du \\ &= \frac{2}{3}u^3 + 4u + c \\ &= \frac{2}{3}u(u^2+6) + c \end{aligned}$$

Now have to replace u by $(x-2)^{\frac{1}{2}}$

So

$$\int \frac{x\,dx}{\sqrt{x-2}} = \frac{2}{3}(x-2)^{\frac{1}{2}}(x-2+6) + c = \frac{2}{3}(x+4)\sqrt{x-2} + c$$

(vii)

$$\int \sin^2 x \cos^3 x \, dx$$

Try substitution

$u = \sin x$

$\Rightarrow \dfrac{du}{dx} = \cos x$

$\Rightarrow \dfrac{dx}{du} = \dfrac{1}{\cos x}$

Therefore

$$\begin{aligned}
\int \sin^2 x \cos^3 x \, dx &= \int \sin^2 x \cos^3 x \frac{dx}{du} \, du \\
&= \int \sin^2 x \cos^2 x \cos x \frac{dx}{du} \, du \\
&= \int \sin^2 x (1 - \sin^2 x) \cos x \frac{dx}{du} \, du \\
&= \int u^2 (1 - u^2) \cos x \frac{1}{\cos x} \, du \\
&= \int (u^2 - u^4) \, du \\
&= \frac{u^3}{3} - \frac{u^5}{5} + c \\
&= \frac{u^3}{15}(5 - 3u^2) + c
\end{aligned}$$

Replace u by $\sin x$ to get:

$$\underline{\int \sin^2 x \cos^3 x \, dx = \frac{1}{15} \sin^3 x (5 - 3 \sin^2 x) + c}$$

(viii)
$$\int_1^2 \frac{8x\,dx}{(2x+1)^3}$$
Try substitution
$u = 2x+1 \Rightarrow x = \frac{1}{2}(u-1)$
$\frac{du}{dx} = 2 \Rightarrow \frac{dx}{du} = \frac{1}{2}$
Under this substitution the interval $1 \leq x \leq 2$ maps onto $3 \leq u \leq 5$ (when $x = 1,\ u = (2 \times 1) + 1 = 3$, when $x = 2,\ u = (2 \times 2) + 1 = 5$)
Hence,
$$\begin{aligned}
& \int \frac{8x}{(2x+1)^3} \\
= & \int_{x=1}^{x=2} \frac{8x}{(2x+1)^4} \frac{dx}{du}\,du \\
= & \int_{u=3}^{u=5} \frac{8 \times \frac{1}{2}(u-1)}{u^3} \times \frac{1}{2}\,du \\
= & 2\int_3^5 \left(\frac{1}{u^2} - \frac{1}{u^3}\right)\,du \\
= & 2\int_3^5 (u^{-2} - u^{-3})\,du \\
= & 2\left[-u^{-1} + \frac{1}{2}u^{-2}\right]_3^5
\end{aligned}$$
NB keep the u ranges in the <u>limits</u> since we have an answer in u
$$\begin{aligned}
= & 2\left\{\left(-\frac{1}{5} + \frac{1}{50}\right) - \left(-\frac{1}{3} + \frac{1}{18}\right)\right\} \\
= & 2\left(-\frac{9}{50} + \frac{5}{18}\right) \\
= & \underline{\frac{44}{225}}
\end{aligned}$$

(ix)

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cot x}{\sqrt{\csc^3 x}}\, dx$$

Let $u = \csc x$

$$\frac{du}{dx} = -\csc x \cot x \Rightarrow \frac{dx}{du} = \frac{-1}{\csc x \cot x}$$

When $x = \frac{\pi}{6},\ u = \csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{(\frac{1}{2})} = 2$

When $x = \frac{\pi}{2},\ u = \csc\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = \frac{1}{1} = 1$

So $\frac{\pi}{6} \le x \le \frac{\pi}{2}$ maps to $2 \le u \le 1$.

Hence:

$$\begin{aligned}
& \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cot x\, dx}{\sqrt{\csc^3 x}} \\
= & \int_{x=\frac{\pi}{6}}^{x=\frac{\pi}{2}} \frac{\cot x}{\sqrt{\csc^3 x}} \frac{dx}{du}\, du \\
= & \int_{u=2}^{u=1} \frac{1}{u^{\frac{3}{2}}} \cot x \left(\frac{-1}{\csc x \cot x}\right) du \\
= & -\int_{u=2}^{u=1} \frac{du}{u^{\frac{3}{2}} \csc x} \\
= & -\int_{u=2}^{u=1} \frac{du}{u^{\frac{3}{2}} \cdot u} \\
= & -\int_{2}^{1} \frac{du}{u^{\frac{5}{2}}} \\
= & +\int_{1}^{2} \frac{du}{u^{\frac{5}{2}}} \\
& \text{since } -\int_a^b f(x)\, dx = +\int_b^a f(x)\, dx \text{ check from definite integral limits} \\
= & \int_1^2 u^{-\frac{5}{2}}\, du \\
= & \left[\frac{-2}{3} u^{-\frac{3}{2}}\right]_1^2 \\
= & \frac{-2}{3}(2^{-\frac{3}{2}} - 1^{-\frac{3}{2}}) \\
= & -\frac{2}{3}\left(\frac{1}{2^{\frac{3}{2}}} - 1\right) \\
= & -\frac{2}{3}\left(\frac{1}{2\sqrt{2}} - 1\right) \\
= & -\frac{2}{3}\left(\frac{\sqrt{2}}{4} - 1\right) \\
= & \underline{\frac{1}{6}(4 - \sqrt{2})}
\end{aligned}$$

(x)

$$\int_0^3 \frac{3\,dx}{\sqrt{9-x^2}}$$

Could use standard integrals <u>or</u> if none available, let

$x = 3\sin u, \quad 0 \le x \le 3 \to 0 \le u \le \frac{\pi}{2}$

$\frac{dx}{du} = 3\cos u$

Hence

$$\begin{aligned}
\int_0^3 \frac{3\,dx}{\sqrt{9-x^2}}\,dx &= \int_0^3 \frac{3}{\sqrt{9-x^2}}\frac{dx}{du}\,du \\
&= \int_0^{\frac{\pi}{2}} \frac{3}{\sqrt{9-9\sin^2 u}} 3\cos u\,du \\
&= \int_0^{\frac{\pi}{2}} \frac{3\cdot 3\cos u}{3\cos u}\,du \\
&= 3\int_0^{\frac{\pi}{2}} du \\
&= 3[u]_0^{\frac{\pi}{2}} \\
&= \underline{\frac{3\pi}{2}}
\end{aligned}$$

(xi)

$$\int \frac{(x-2)\,dx}{(x^2-4x-5)}$$

Use partial fractions since it's of form $\dfrac{\alpha x+\beta}{\gamma x^2+\delta x+\epsilon}$

First we have $x^2-4x-5 \equiv (x+1)(x-5)$ factors: $(x+1),(x-5)$

So we seek $A,\ B$ such that

$$\begin{aligned}
\frac{x-2}{(x+1)(x-5)} &\equiv \frac{A}{(x+1)} + \frac{B}{(x-5)} \\
&\Rightarrow (x-2) \equiv A(x-5) + B(x+1)
\end{aligned}$$

Equating coefficients of x and numbers on LHS and RHS $\Rightarrow 1 = A+B, -2 = -5A+B$

Solving gives $A = \frac{1}{2}, B = \frac{1}{2}$

$$\begin{aligned}
\int \frac{(x-2)\,dx}{(x^2-4x-5)} &= \frac{1}{2}\int \frac{dx}{(x+1} + \frac{1}{2}\int dx(x-5) \\
&= \frac{1}{2}\log(x+1) + \frac{1}{2}\log(x-5) + c \\
\text{So} \qquad &= \frac{1}{2}\log[(x+1)(x-5)] + c \\
&= \underline{\frac{1}{2}\log(x^2-4x-5) + c} \\
\text{or} \quad &= \log(\sqrt{x^2-4x-5}) + c \\
\text{or} \quad &= \log(k\sqrt{x^2-4x-5}) + c
\end{aligned}$$

k = arbitrary constant

<u>NB</u> could also do this using the $\dfrac{f'(x)}{f(x)}$ method since

$$\int \frac{x-2}{(x^2-4x-5)} = \frac{1}{2}\int \frac{2x-4}{x^2-4x-5} = \frac{1}{2}\int \frac{f'(x)}{f(x)} \text{ where } f(x) = x^2-4x-5$$

Check for yourself that the answers are identical (up to arbitrary constants).

(xii)

$$\int \frac{dx}{(x^2+6x+17)}$$

Now $x^2+6x+17 \equiv (x+3)^2+8$

So

$$\begin{aligned}
\int \frac{dx}{(x^2+6x+17)} &= \int \frac{dx}{(x+3)^2+(\sqrt{3})^2} \\
&\quad \text{Now set u} = \text{x} + 3,\ \text{du} = \text{dx} \\
&= \int \frac{du}{u^2+a^2} \text{ where a} = \sqrt{3} \\
&= \frac{1}{a}\arctan\left(\frac{u}{a}\right) + c \text{ by standard integral} \\
&= \underline{\frac{1}{\sqrt{8}}\arctan\left(\frac{x+3}{\sqrt{8}}\right) + c}
\end{aligned}$$

NB Could you have used partial fractions here?

(xiii)

$$\int \frac{(2x+5)\,dx}{(x^2+4x+5)}$$

Could try partial fractions or spot type $\dfrac{f'(x)}{f(x)}$: do this way.

$$\frac{d}{dx}(x^2+4x+5) = 2x+4$$

Hence

$$\begin{aligned}
& \int \frac{2x+5}{x^2+4x+5}\,dx \\
&= \int \frac{(2x+4)\,dx}{x^2+4x+5} + \int \frac{1\,dx}{x^2+4x+5} \\
&= \log(x^2+4x+5) + \int \frac{dx}{x^2+4x+5} \qquad \text{(standard integrals)} \\
&\quad \ln\left(\frac{f'}{f}\right) \text{ type integral and integral of type (xxiv)} \\
&= \log(x^2+4x+5) + \int \frac{dx}{(x+2)^2+1^2} \\
&= \underline{\log(x^2+4x+5) + \arctan(x+2) + c}
\end{aligned}$$

(xiv)
$\int \frac{x^2\,dx}{(x+1)}$
Partial fractions won't work: divide out.

$$\frac{x^2}{x+1} \equiv x - 1 + \frac{1}{x+1}$$

This comes either from
PICTURE

$$\Rightarrow \frac{x^2}{x+1} = (x-1) + \left(\frac{1}{x+1}\right)$$

$$\begin{aligned} \frac{x^2}{x+1} = \frac{x^2-1+1}{x+1} &= \frac{x^2-1}{x+1} + \frac{1}{x+1} \\ \text{\underline{or}} \qquad &= \frac{(x-1)(x+1)}{(x+1)} + \frac{1}{x+1} \\ &= (x-1) + \frac{1}{x+1} \end{aligned}$$

Thus $\int \frac{x^2}{x+1}\,dx = \int 9x - 1)\,dx + \int \frac{dx}{(x+1)} = \underline{\frac{x^2}{2} - x + \log(x+1) + c}$