QUESTION

This question has been broken down into a series of cases. However, the objective is to prove that $\phi(n) \leq n - \sqrt{(n)}$ when n is a composite integer. Therefore full marks for parts (i)-(vii) may alternatively be obtained by just giving an alternative proof for part (vii).

(i) Give, without proof, the formula for Euler's function, $\phi(n)$, in terms of the prime power factorisation of n.

(ii) If p_1, p_2 are distinct primes show that

$$\phi(p_1p_2) \leq p_1p_2 - \sqrt{p_1p_2}.$$

(iii) If p_1, p_2 are distinct primes and α is an integer such that $\alpha \geq 2$ show that

$$\phi(p_1p_2^\alpha) \leq p_1p_2^\alpha - \sqrt{p_1p_2^\alpha}.$$

(iv) If p_1, p_2, p_3 are distinct primes show that

$$\phi(p_1p_2p_3) \leq p_1p_2p_3 - \sqrt{p_1p_2p_3}.$$

(v) If p is a prime and α is a positive integer such $\alpha \geq 2$ show that

$$\phi(p^\alpha) \leq p^\alpha - \sqrt{p^\alpha}.$$

(vi) If $n_1, n_2 \geq 2$ are integers show that

$$(n_1 - \sqrt{n_1})(n_2 - \sqrt{n_2}) \leq (n_1n_2 - \sqrt{n_1n_2}).$$

(vii) If n is an integer which is not prime use parts (i)-(vi) to show that

$$\phi(n) \leq n - \sqrt{n}.$$

(viii) Does the inequality of (vi) hold when n is prime?

ANSWER

(i) If $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ with each $\alpha_i \geq 1$ and $p_1, \dots, p_r$ distinct primes then

$$\phi(n) = p_1^{\alpha_1-1}(p_I - 1)p_2^{\alpha_2-1}(p_2-1)\dots p_r^{\alpha_r-1}(p_r-1).$$

(ii) If $2 \leq p_1 < p_2$ then $\phi(p_1p_2)p_1p_2 - p_1 - p_2 + 1$ so that the required inequality will follow from

$$\sqrt{p_1p_2} \leq p_1 + p_2 - 1$$

which follows in turn from

$$p_1p_2 \leq p_2^3 \leq (p_i + p_2 - 1)^2.$$

(iii) We have

$$\begin{aligned}
\phi(p_1p_2^\alpha) &= (p_1 - 1)(p_2^\alpha - p_2^{\alpha-1}) \\
&= p_1p_2^\alpha - p_2^\alpha - p_1p_2^{\alpha 1} + p_2^{\alpha-1} \\
&\leq p_1p_2^\alpha - p_1p_2^{\alpha-1} \\
&\leq p_1p_2^\alpha - \sqrt{p_1p_2^\alpha}
\end{aligned}$$

since $\sqrt{p_1} < p_1$ and $\frac{\alpha}{2} \leq \alpha - 1$.

(iv) If $2 \leq p_1 < p_2 < p_3$ then

$$\phi(p_1p_2p_3) = p_1p_2p_3 - p_1p_2 - p_1p_3 - p_2p_3 + p_1 + p_2 + p_3 - 1$$

so that the required inequality will follow from

$$\sqrt{p_1p_2p_3} \leq p_1p_2 + p_1p_3 + p_2p_3 - p_1 - p_2 - p_3 + 1$$

which follows in turn from

$$p_1p_2p_3 \leq (p_1p_3 + p_1p_2 - 1)^2 \leq (p_1p_2 + p_1p_3 + p_2p_3 - p_1p_2p_3 + 1)^2$$

which holds because $p_2p_3 \geq 3p_3 \geq p_1 + p_2 + p_3$.

(v) We have $\phi(p^\alpha) = p^\alpha - p^{\alpha-1} \leq p^\alpha - \sqrt{p^\alpha}$ since $\alpha - 1 \geq \frac{\alpha}{2}$.

(vi) We have

$$\begin{aligned}
(n_1 - \sqrt{n_1})(n_2 - \sqrt{n_2}) &= n_1n_2 - \sqrt{n_1n_2} + \sqrt{n_1}(\sqrt{n_2} - n_2) + \sqrt{n_2}(\sqrt{n_1} - n_1) \\
&\leq n_1n_2 - \sqrt{n_1n_2}
\end{aligned}$$

(vii) Suppose that the prime factorisation of n is $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$. Since n is composite we may write $n = n_1 n_2 \dots n_s$ where $n_1, \dots, n_s$ are pairwise coprime positive integers each of which is one of the type considered in parts (ii)-(v). The result now follows from part (vi) since

$$\begin{aligned} \phi(n) &= \phi(n_1) \dots \phi(n_s) \\ &\leq (n_1 - \sqrt{n_1}) \dots (n_s - \sqrt{n_s}) \\ &\leq n_1 \dots n_s - \sqrt{n_1 \dots n_s} \end{aligned}$$

by induction on s.

(viii) When n is prime then $\phi(n) = n - 1$ which is greater than $n - \sqrt{n}$ because $1 < \sqrt{n}$. Therefore the inequality fails fro all primes.