QUESTION

(i) State the Law of Quadratic Reciprocity.

(ii) Use (i) to evaluate the Legendre symbol $\left(\frac{3}{p}\right)$ when p is an odd prime. More precisely, show that

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \text{ (modulo 12)}, \\ -1 & \text{if } p \equiv \pm 5 \text{ (modulo 12)}. \end{cases}$$

ANSWER

(i) The Law of Quadratic Reciprocity states that, if p and q are distinct odd primes then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}$$

(ii) Therefore

$$\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\frac{(p-1)}{2}}$$

and

$$\left(\frac{p}{3}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \text{ (modulo 3)}, \\ -1 & \text{if } p \equiv -1 \text{ (modulo 3)}. \end{cases}$$

Also $p = 3k \pm 1$ can only happen if $k = 2s$, since p is odd. Hence we write $p = 6s \pm 1$. The possibilities modulo 12 for p are $12t - 1, 12 + 1, 12t + 5, 12 + 7$ which we shall deal with case by case.

If $p = 12 - 1$ then

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right)(-1)^{\frac{(p-1)}{2}} = (-1)(-1)^{(6t-1)} = 1.$$

If $p = 12t + 1$ then

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right)(-1)^{\frac{(p-1)}{2}} = (-1)^{(6t)} = 1.$$

If $p = 12t + 5$ then

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right)(-1)^{\frac{(p-1)}{2}} = (-1)^{(6t+3)} = -1,$$

as required.