QUESTION

Find all the solutions of each of the following congruences, expressing your answers in terms of congruence classes, $[x]$:

(i) $7x \equiv 29$ (modulo 64),

(ii) $3x^5 + x^3 + 3001 \equiv 9$ (modulo 14).

ANSWER

(i) Since 7 is odd, the residue class [7] represent a unit modulo 64. Hence we need only find the unique $[y] \in U_{64}$ such that $[x][y] = [1]$ and then the unique solution to (1) is $[x] = [29y] \in U_{64}$. Since $7^2 = 49 = 3\times16+1$ we see that $y = 16k + 7$ for some integer k. Now consider the congruence

$$7 \times (16K + 7) = 1 + 16(7k + 3) \text{ (modulo 64).}$$

If we choose $7k+3$ to be a multiple of 4 then we have a suitable solution for y. The smallest convenient value is $k = 3$ and $y = 55$. Hence the only solution is $x \equiv 29 \times 55 \equiv 59$ (modulo 64) of $[x] = [59] \in U_{64}$.

(ii) We begin by finding all the solutions to this congruence modulo 2 and 7. Modulo 2 all x will yield a solution, Modulo 7 there is no solution if $x \equiv 0$ (modulo 7) since $3001 \equiv 201 \equiv 61 \equiv 5$ (Modulo7). If $x = 1, 2, 4$ then $x^3 \equiv 1$ (modulo 7) while if $x = 3, 5, 6$ then $x^3 \equiv -1$ (modulo 7). Therefore, if $x = 1, 2, 4$, we are trying to solve $3x^5 \equiv 9-5-1 \equiv 2-6 \equiv 3$ (modulo 7). On the other hand $x^6 \equiv 1$ (modulo 7) so that the solutions to this are $3 \equiv 3x^6 \equiv 3x^5x \equiv 3x$ (modulo 7) or $x \equiv 1$ (modulo 7). On the other hand, if $x = 3, 5, 6$ a similar argument shows that $3x^5 \equiv 9 - 5 + 1 \equiv 2 - 4 \equiv 5$ (modulo 7). Multiplying by x yields $3 \equiv 3x^6 \equiv 3x^5x \equiv 5x$ (modulo 7). The solution to this is $x \equiv 2$ (modulo 7) which is not one of $x = 3, 5, 6$ so there is no such solution.

Hence we have shown that all the solutions must lie in the set of integers $\{1+7k|k \text{ an integer}\}$ which give the two residue classes [1], [8] (modulo 14).