Question

Find the inverses of the following matrices using cofactors:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 6 \end{pmatrix}$$

Answer

Inverse of $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

det(a) = 1

$A^{-1} = \frac{adj(A)}{|A|}$

Matrix of cofactors

$$[A_{ij}] \quad = \left[\begin{array}{cccc} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \\ & & & \\ -\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \\ & & & \\ \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\ & & & \\ -\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \end{array}\right]$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

Hence $A^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$

Finding the inverse of $B = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 6 \end{pmatrix}$

Matrix of cofactors

$$
\begin{aligned}
[A_{ij}] &= \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 1 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \\ -\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \\ \\ \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \end{bmatrix} \\
&= \begin{pmatrix} 4 & -2 & -1 \\ -3 & 6 & 0 \\ -1 & -1 & 1 \end{pmatrix}
\end{aligned}
$$

Hence $B^{-1} = \frac{1}{3} \begin{pmatrix} 4 & -3 & -1 \\ -2 & 6 & -1 \\ -1 & 0 & 1 \end{pmatrix}$