Question
Solve the equations:

$$\begin{aligned} x + y + z &= 2 \\ y + 2z &= 0 \\ 2x - y - z &= 1 \end{aligned}$$

(a) by rewriting the equation in the form $\mathbf{A}\mathbf{x} = \mathbf{b}$ and finding $\mathbf{A}^{-1}$ by the cofactor method

(b) using the standard elimination method.

Which method took longer?

Answer

(a) Equations can be rewritten as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & -1 \end{vmatrix} = 3 \times (-1 + 2) = 3$$

Call matrix of cofactors a

$$a = \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 4 & -2 \\ 0 & -3 & 3 \\ 1 & -2 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{a^T}{|A|} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{4}{3} & -1 & \frac{-2}{3} \\ \frac{-2}{3} & 1 & \frac{1}{3} \end{pmatrix}$$

$$\text{Finally } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{4}{3} & -1 & \frac{-2}{3} \\ \frac{-2}{3} & 1 & \frac{1}{3} \end{pmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

(b) elimination

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \rightarrow (\text{row } 3 \rightarrow \text{row } 3 + 2\text{row } 1)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \rightarrow (\text{row } 3 \rightarrow \text{row } 3 + 3\text{row } 2)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

$$\text{Hence} \quad \begin{array}{rcr} x + y + z & = & 2 \\ y + 2z & = & 0 \\ 3z & = & -3 \end{array} \Rightarrow \begin{array}{l} z = -1 \\ y = -2z = 2, \\ x = 2 - y - z = 1 \end{array}$$

Cofactor way is MUCH longer.