Question
Use the elimination method to find the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -1 & 3 & 2 \end{pmatrix}$$

Check that the answer obeys the relations $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

Answer
Elimination method to find inverse of A.

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{array}{l} \\ \text{row } 2 \to \text{row } 2 - \text{row } 1 \\ \text{row } 3 \to \text{row } 3 + \text{row } 1 \end{array}$$

$$\to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{array}{l} \text{row } 1 \to \text{row } 1 + \frac{2}{3}\text{row } 2 \\ \\ \text{row } 3 \to \text{row } 3 + \frac{5}{3}\text{row } 1 \end{array}$$

$$\to \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & -3 & 2 \\ 0 & 0 & \frac{13}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ -1 & 1 & 0 \\ -\frac{2}{3} & \frac{5}{3} & 1 \end{pmatrix} \quad \begin{array}{l} \text{row } 1 \to \text{row } 1 - \frac{1}{13}\text{row } 3 \\ \text{row } 2 \to \text{row } 2 - \frac{6}{13}\text{row } 3 \\ \\ \end{array}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & \frac{13}{3} \end{pmatrix} \begin{pmatrix} \frac{5}{13} & \frac{7}{13} & -\frac{1}{13} \\ -\frac{9}{13} & \frac{3}{13} & -\frac{6}{13} \\ -\frac{2}{3} & \frac{5}{3} & 1 \end{pmatrix} \quad \begin{array}{l} \\ \text{row } 2 \to -\frac{1}{3}\text{row } 2 \\ \text{row } 3 \to \frac{3}{13}\text{row } 3 \end{array}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{13} & \frac{7}{13} & -\frac{1}{13} \\ -\frac{3}{13} & -\frac{1}{13} & \frac{2}{13} \\ -\frac{2}{13} & \frac{5}{13} & \frac{3}{13} \end{pmatrix}$$

To check that this is the inverse check that $AA^{-1} = I$ and $A^{-1}A = I$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -1 & 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{5}{13} & \frac{7}{13} & -\frac{1}{13} \\ -\frac{3}{13} & -\frac{1}{13} & \frac{2}{13} \\ -\frac{2}{13} & \frac{5}{13} & \frac{3}{13} \end{pmatrix}$$

$$= \frac{1}{13} \begin{pmatrix} 5+6+2 & 7-2-5 & -1+4-3 \\ 5-3-2 & 7+1+5 & -1-2+3 \\ -5+9-4 & -7-3+10 & 1+6+6 \end{pmatrix} = I$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -1 & 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{5}{13} & \frac{7}{13} & -\frac{1}{13} \\ -\frac{3}{13} & -\frac{1}{13} & \frac{2}{13} \\ -\frac{2}{13} & \frac{5}{13} & \frac{3}{13} \end{pmatrix}$$

$$= \frac{1}{13} \begin{pmatrix} 5+7+1 & 10-7-3 & -5+7-2 \\ 3-1-2 & 6+1+6 & -3-1+4 \\ -2+5-3 & -4-5+9 & 2+5+6 \end{pmatrix} = I$$

Hence $A^{-1} = \begin{pmatrix} \frac{5}{13} & \frac{7}{13} & -\frac{1}{13} \\ -\frac{3}{13} & -\frac{1}{13} & \frac{2}{13} \\ -\frac{2}{13} & \frac{5}{13} & \frac{3}{13} \end{pmatrix}$

and $AA^{-1} = A^{-1}A = I$ as required.