Question

Show that if E is a subset of $\mathbf{R}$ with finite measure, and if the function $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = m^*(E \cap (-\infty, x))$ then $f(x)$ is continuous. Can you find similar theorems in $\mathbf{R}^2$, or $\mathbf{R}^n$?

Answer

$x > a \Rightarrow 0 \geq f(x) - f(a) = m^*(E \cap (-\infty, x)) - m^*(E \cap (-\infty, a))$
$\leq m^*(E \cap [x, a)) \leq m^*([x, a)) = x - a$
Similarly $x < a \Rightarrow 0 \leq f(x) - f(a) \leq a - x$
Hence continuity.
$\mathbf{R^2}$ by lines.
$\mathbf{R^3}$ by planes.