Question

The four points A, B, C, D have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

(a) Show that the four points A, B, C, D are coplanar if and only if

$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} + \delta\mathbf{d} = \mathbf{0}$ and $\alpha + \beta + \gamma + \delta = 0$ with not all $\alpha, \beta, \gamma, \delta$ zero.

(b) What conditions must be placed on $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ such that A, B, C, D form

(i) a parallelogram

(ii) a tetrahedron.

(c) When $\mathbf{d} = \mathbf{0}$ find the position vector of intersection of

(i) the diagonals of the parallelogram $ABCD$

(ii) the lines joining the midpoints of opposite edges of the tetrahedron $ABCD$.

Answer

(a) $ABCD$ are coplanar if and only if $\vec{AD} = k\vec{AB} + l\vec{AC}$ for some k, l

if and only if $\mathbf{d} - \mathbf{a} = k(\mathbf{b} - \mathbf{a}) + l(\mathbf{c} - \mathbf{a})$

if and only if $\mathbf{a}(k + l - 1) + \mathbf{b}(-k) + \mathbf{c}(-l) + \mathbf{d} = 0$

$\Rightarrow \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} + \delta\mathbf{d} = 0$ and $\alpha + \beta + \gamma + \delta = 0$

Conversely if $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} + \delta\mathbf{d} = 0$ and $\alpha + \beta + \gamma + \delta = 0$

If $\delta = 0$

$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0$ and $\alpha + \beta + \gamma = 0$ So A, B, C are colinear

Thus $ABCD$ are coplanar.

If $\delta \neq 0$

$\frac{\alpha}{\delta}\mathbf{a} + \frac{\beta}{\delta}\mathbf{b} + \frac{\gamma}{\delta}\mathbf{c} + \mathbf{d} = 0$

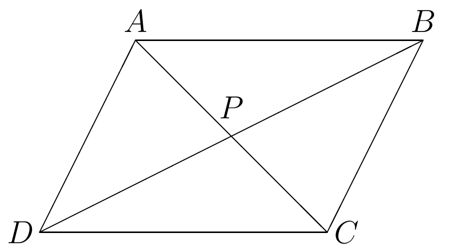
Let $\frac{\beta}{\delta} = -k \quad \frac{\gamma}{\delta} = -l$ then $\frac{\alpha}{\delta} = k + l - 1$

Hence $ABCD$ are coplanar.

(b) $ABCD$ form a parallelogram in that order if $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$ with none of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, equal.

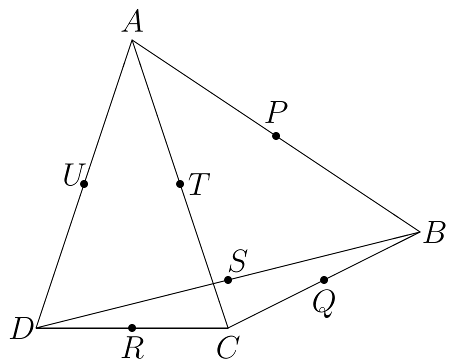
$ABCD$ form a tetrahedron if the are not coplanar.

(c) **(i)**



$P = \frac{1}{2}\mathbf{b}$

(ii)



$\mathbf{S} = \frac{1}{2}\mathbf{b}$ $\qquad t = \frac{1}{2}(\mathbf{a} + \mathbf{c})$

The mid point of ST is

$$\frac{1}{2}\left(\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c}\right) = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$$

which lies on the other lines by symmetry.