Question
Given that $\mathbf{a}$ is an arbitary vector and $\mathbf{e}$ is a fixed unit vector, determine the magnitude and direction of $\mathbf{e} \times (\mathbf{a} \times \mathbf{e})$ and deduce that

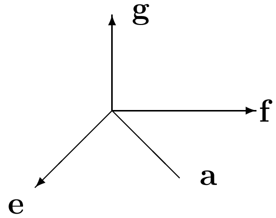
$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{e})\mathbf{e} + \mathbf{e} \times (\mathbf{a} \times \mathbf{e})$$

explaining this result.

Consider any three non-collinear vectors a, b, c. By resolving $\mathbf{a}$ along $\mathbf{b}, \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$ use the above result to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{c} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}$$

Answer



$\mathbf{a} = \mathbf{a} \cdot \mathbf{e}\,\mathbf{e} + \mathbf{a} \cdot \mathbf{f}\,\mathbf{f}$
So $\mathbf{a} \times \mathbf{e} = (\mathbf{a} \cdot \mathbf{f})\mathbf{f} \times \mathbf{e} = -(\mathbf{a} \cdot \mathbf{f})\mathbf{g}$
Thus $\mathbf{e} \times (\mathbf{a} \times \mathbf{e}) = -(\mathbf{a} \cdot \mathbf{f})\mathbf{e} \times \mathbf{g} - (\mathbf{a} \cdot \mathbf{f})\mathbf{f}$
So $\mathbf{a} = \beta\mathbf{b} + \gamma\mathbf{c} + \delta(\mathbf{b} \times \mathbf{c})$

$$\begin{aligned} a \times (b \times c) &= \beta b \times (b \times c) + \gamma(c \times (b \times c)) \\ &= \beta b^2[(\hat{b} \cdot c)\hat{b} - \mathbf{c}] + \gamma c^2[\mathbf{b} - (\hat{c} \cdot b)\hat{c}] \\ &= \beta(b \cdot c)\mathbf{b} + \gamma(c \cdot c)\mathbf{b} - (\beta(b \cdot b)\mathbf{c} + \gamma(c \cdot b)\mathbf{c}) \end{aligned}$$

Now

$$\begin{aligned} a \cdot c &= \beta(b \cdot c) + \gamma(c \cdot c) \quad \text{since } (b \times c) \cdot c = 0 \\ a \cdot b &= \beta(b \cdot b) + \gamma(c \cdot b) \\ \text{So} \quad a \times (b \times c) &= (a \cdot c)\mathbf{b} - (a \cdot b)\mathbf{c} \end{aligned}$$