Question
When $k\mathbf{x} + (\mathbf{x}\cdot\mathbf{a})\mathbf{b} = \mathbf{c}$ $(b \neq 0)$ explain why $\mathbf{x}$ is of the form $\frac{1}{k}\mathbf{c} + \alpha\mathbf{b},\ k \neq 0\ \alpha\epsilon\mathbf{R}$. Hence solve $k\mathbf{x} + (\mathbf{x}\cdot\mathbf{a})\mathbf{b} = \mathbf{c}$

(i) $k + a\cdot b \neq 0$

(ii) $k + a\cdot b = 0$ and $\mathbf{a}\cdot\mathbf{c} = 0$ or $\mathbf{a}\cdot\mathbf{c} \neq 0$.

What can you say about possible solutions if $k = 0$?

Answer
If $k\mathbf{x} + (\mathbf{x}\cdot\mathbf{a})\mathbf{b} = \mathbf{c}$
Then $\mathbf{x} = \frac{1}{k}\mathbf{c} - \frac{1}{k}(\mathbf{x}\cdot\mathbf{a})\mathbf{b} = \frac{1}{k}\mathbf{c} + \alpha\mathbf{b} \quad k \neq 0\ \ \alpha\epsilon\mathbf{R}$
So substituting for $\mathbf{x}$ in the original equation gives
$k\mathbf{\frac{1}{k}}\mathbf{c} + \alpha\mathbf{b} + (\mathbf{\frac{1}{k}}\mathbf{c} + \alpha\mathbf{b}\cdot\mathbf{a})\mathbf{b} = \mathbf{c}$
$(k\alpha + \frac{1}{k}\mathbf{c}\cdot\mathbf{a} + \alpha\mathbf{b}\cdot\mathbf{a})\mathbf{b} = \mathbf{0}$
So $\alpha(k + \mathbf{b}\cdot\mathbf{a} = -\frac{1}{k}(c\cdot a) \quad \mathbf{b} \neq \mathbf{0}$

(i) So $\alpha = -\dfrac{1}{k}\dfrac{\mathbf{c}\cdot\mathbf{a}}{k + \mathbf{b}\cdot\mathbf{a}}$ if $k + \mathbf{b}\cdot\mathbf{a} \neq 0$

Thus in this case we have

$$x = \frac{1}{k}\mathbf{c} - \frac{1}{k}\frac{\mathbf{c}\cdot\mathbf{a}}{k + \mathbf{b}\cdot\mathbf{a}}\mathbf{b}$$

(ii) If $k + \mathbf{b}\cdot\mathbf{a} = 0$ and $\mathbf{c}\cdot\mathbf{a} = 0$ then any α will give a solution.

If $k + \mathbf{b}\cdot\mathbf{a} = 0$ and $\mathbf{c}\cdot\mathbf{a} = 0$ then are no solutions.

If $k = 0$ then the equation reduced to $(x\cdot a)\mathbf{b} = \mathbf{c}$

If $\mathbf{b} = \mathbf{c} = 0$ all $\mathbf{x}$ are solutions.

If only $\mathbf{b} = 0$ then there are no solution.

If $\mathbf{c} = 0$, $\mathbf{b} \neq 0$ then $\mathbf{x}\cdot\mathbf{a} = 0$ so $\mathbf{x}$ and $\mathbf{a}$ are perpendicular.

If $\mathbf{c} \neq 0$, $\mathbf{b} \neq 0$ there are no solutions so $\mathbf{b}$ and $\mathbf{c}$ are not parallel.

If $\mathbf{b}$ and $\mathbf{c}$ are parallel then $|(x\cdot a)| = \dfrac{|c|}{|b|}$ i.e. $(x\cdot a) = \pm\frac{|c|}{|b|}$ as $\mathbf{b}$ and $\mathbf{c}$ have the same or opposite direction. Thus $\mathbf{x} = \pm\frac{|c|}{|b||a|^2}\mathbf{a} + \mathbf{b}$ where $\mathbf{b}$ and $\mathbf{a}$ are parallel.