Question

(a) Show that the plane through the point $\mathbf{a}, \mathbf{b}, \mathbf{c}$ has the equation

$$\mathbf{r} \cdot \mathbf{b} \times \mathbf{c} + \mathbf{r} \cdot \mathbf{c} \times \mathbf{a} + \mathbf{r} \cdot \mathbf{a} \times \mathbf{b} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

(b) Prove that the shortest distance from the point $\mathbf{a}$ to the line joining the points $\mathbf{b}$ and $\mathbf{c}$ is given by

$$\frac{|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|}{|\mathbf{b} - \mathbf{c}|}$$

Answer

(a) if the plane passes through the points $\mathbf{b}$, $\mathbf{c}$ it has a normal vector

$$\begin{aligned} (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) &= b \times c - a \times c - b \times a \\ &= b \times c + c \times a + a \times b \end{aligned}$$

Thus the equation of the plane is

$$\mathbf{r}(b \times c + c \times a + a \times b) = k$$

Now $\mathbf{r} = \mathbf{a}$ lies in the plane and so

$$k = \mathbf{a}(b \times c + c \times a + a \times b) = a \cdot b \times c$$

therefore the equation is

$$\mathbf{r} \cdot (b \times c + c \times a + a \times b) = a \cdot b \times c$$

(b) If a line has equation $r = \mathbf{s} + t\mathbf{u}$ the shortest distance of $\mathbf{p}$ form the line is $\frac{|(\mathbf{s} - \mathbf{p}) \times \mathbf{u}|}{|u|}$. The equation of the line here is $\mathbf{r} = \mathbf{b} + t(\mathbf{c} - \mathbf{b})$

So the shortest distance of $\mathbf{a}$ from the line is

$$\frac{|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b})|}{|\mathbf{c} - \mathbf{b}|} = \frac{|b \times c + c \times a + a \times b|}{|\mathbf{c} - \mathbf{b}|}$$