Question

(a) Obtain the value(s) of λ which make the vectors $\mathbf{i}-\mathbf{j}$, $2\mathbf{i}+\mathbf{j}-\lambda\mathbf{k}$, $\lambda\mathbf{i}-\mathbf{j}+\lambda\mathbf{k}$ coplanar.

(b) Show that the four points $(5,2-1)$, $(6,1,4)$, $(-1,-3,6)$, $(-3,-2,-1)$ lie in a plane.

Answer

(a) The three vectors $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$ are coplanar if $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=0$

$$\begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & -\lambda \\ \lambda & -1 & \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & -\lambda \\ \lambda & \lambda-1 & \lambda \end{vmatrix} = 0$$
$$\Rightarrow \quad 3\lambda+\lambda^2-\lambda=0$$
$$\Rightarrow \quad \lambda^2+2\lambda=0$$
$$\Rightarrow \quad \lambda=0, \text{ or } \lambda=-2$$

(b) The 4 points all lie in a plane if there exist α, β, γ, δ are not all zero such that $\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{d}=0$ and $\alpha+\beta+\gamma+\delta=0$

$$\begin{aligned} 5\alpha+6\beta-2\gamma-3\delta &= 0 & (1)\\ 2\alpha+\beta-3\gamma-2\delta &= 0 & (2)\\ -3\alpha+4\beta+6\gamma-\delta &= 0 & (3)\\ \alpha+\beta+\gamma+\delta &= 0 & (4) \end{aligned}$$

Eliminating δ from equations (1) to (3) using equation (4) gives:

$$\begin{aligned} 8\alpha+9\beta+\gamma &= 0 & (5)\\ 4\alpha+3\beta-\gamma &= 0 & (6)\\ -2\alpha+5\beta-3\gamma-\delta &= 0 & (7) \end{aligned}$$

Eliminating γ from equations (5) and (7) using equation (6) gives:

$$\begin{aligned} 12\alpha + 129\beta &= 0 \\ 26\alpha + 16\beta &= 0 \end{aligned}$$

[or use the triple product]

So

$$\beta = -\alpha \qquad \gamma = \alpha \qquad \delta = -\gamma$$

so they lie in a plane.