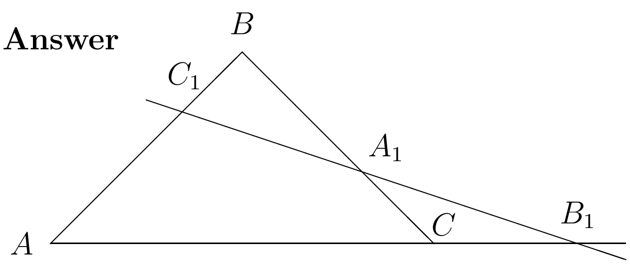
Question

Prove Menelanou's theorem: "If a line cuts the sides of a triangle ABC in the points C_1, A_1, B_1 then $\dfrac{AC_1}{C_1B} \cdot \dfrac{BA_1}{A_1C} \cdot \dfrac{CB_1}{B_1A} = -1$".
Show, conversely, that if the above product of the ratios is -1 then C_1, A_1, B_1 are colinear.

Answer



Let A, B, C, have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,
Then $\mathbf{c}_1 = \dfrac{\mu_1\mathbf{a} + \lambda_1\mathbf{b}}{\lambda_1 + \mu_1}$ $\quad \mathbf{a}_1 = \dfrac{\mu_2\mathbf{b} + \lambda_2\mathbf{c}}{\lambda_2 + \mu_2}$ $\quad \mathbf{b}_1 = \dfrac{\mu_3\mathbf{c} + \lambda_3\mathbf{a}}{\lambda_3 + \mu_3}$

$C_1A_1B_1$ are collinear if and only if for some $t\epsilon\mathbf{R}$, $\mathbf{b}_1 = t\mathbf{c}_1 + (1-t)\mathbf{a}_1$
Thus

$$\frac{\mu_3\mathbf{c} + \lambda_3\mathbf{a}}{\lambda_3 + \mu_3} = t\frac{\mu_1\mathbf{a} + \lambda_1\mathbf{b}}{\lambda_1 + \mu_1} + (1-t)\frac{\mu_2\mathbf{b} + \lambda_2\mathbf{c}}{\lambda_2 + \mu_2}$$

i.e. iff

$$\left.\begin{array}{rcl} \dfrac{\mu_3}{\lambda_3 + \mu_3} &=& (1-t)\dfrac{\lambda_2}{\lambda_2 + \mu_2} \\ 0 &=& t\dfrac{\lambda_1}{\lambda_1 + \mu_1} + (1-t)\dfrac{\mu_2}{\lambda_2 + \mu_2} \\ \dfrac{\lambda_3}{\lambda_3 + \mu_3} &=& t\dfrac{\mu_1}{\lambda_1 + \mu_1} \end{array}\right\} \text{Comparing of } \mathbf{a}, \mathbf{b}, \mathbf{c} \text{ independently.}$$

From the third and first equations, substituting in the second we have,

$$\begin{array}{rcl} 0 &=& \dfrac{\lambda_3}{\lambda_3 + \mu_3} \cdot \dfrac{\lambda_1 + \mu_1}{\mu_1} \cdot \dfrac{\lambda_1}{\lambda_1 + \mu_1} + \dfrac{\mu_3}{\lambda_3 + \mu_3} \cdot \dfrac{\lambda_2 + \mu_2}{\lambda_2} \cdot \dfrac{\mu_2}{\lambda_2 + \mu_2} \\ &=& \dfrac{\lambda_3\lambda_1}{\mu_1} + \dfrac{\mu_3\mu_2}{\lambda_2} \\ -1 &=& \dfrac{\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3} \end{array}$$