Question

(a) If $\mathbf{x}\cdot\mathbf{x} = \alpha^2$ show that the length of $\mathbf{x}$ is fixed but its direction is arbitrary. Hence solve for $\mathbf{x}$

$$a\mathbf{x}\cdot\mathbf{x} + 2\mathbf{b}\cdot\mathbf{x} + c = 0$$

(b) Eliminate $\mathbf{y}$ from the equations

$$\begin{aligned} p\mathbf{x} - \mathbf{a}\times\mathbf{y} &= \mathbf{c} \\ \mathbf{b}\times\mathbf{x} + \mathbf{y} &= \mathbf{0} \end{aligned}$$

Hence solve that this pair of simultaneous equations for $\mathbf{x}$ and $\mathbf{y}$ when $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent, considering the 3 cases $p \neq \mathbf{a}\cdot\mathbf{b}$, $p = \mathbf{a}\cdot\mathbf{b}, p = 0$.

Answer

(a) If $x\cdot x = \alpha^2$ then $|x| = |\alpha|$ so the magnitude of x is fixed.

$$\begin{aligned} a\left(x\cdot x + \frac{2}{a}b\cdot x + \frac{c}{a}\right) &= 0 \\ \left(\mathbf{x} + \frac{b}{a}\right)\cdot\left(\mathbf{x} + \frac{b}{a}\right) &= \frac{b^2 - ac}{a^2} \end{aligned}$$

So

$$x = \frac{1}{a}\mathbf{b} + \sqrt{\frac{b^2 - ac}{a^2}}\hat{e}$$

where $\hat{e}$ is an arbitrary unit vector provided $b^2 \geq ac$

(b)

$$\begin{aligned} p\mathbf{x} - \mathbf{a}\times\mathbf{y} &= \mathbf{c} \\ \mathbf{b}\times\mathbf{x} + \mathbf{y} &= \mathbf{0} \\ \text{So} \quad p\mathbf{x} + \mathbf{a}\times(\mathbf{b}\times\mathbf{x}) &= \mathbf{c} \\ p\mathbf{x} + (a\cdot x)\mathbf{b} - (a\cdot b)\mathbf{x} &= \mathbf{c} \\ \text{or} \quad (p - a\cdot b)\mathbf{x} + (a\cdot x)\mathbf{b} &= \mathbf{c} \quad (*) \end{aligned}$$

This is the same equation as question 15.

If $p - \mathbf{a}\cdot\mathbf{b} \neq 0$ then $\mathbf{x} = \dfrac{c - (a\cdot x)\mathbf{b}}{p - a\cdot b}$

Take the scalar product of (*) with $\mathbf{b}$ to find $(a\cdot x) = \dfrac{a\cdot c}{p}\ \ p \neq 0$

So

$$x = \frac{\mathbf{c} - \frac{a\cdot c}{p}\mathbf{b}}{p - \mathbf{a}\cdot\mathbf{b}}$$

If $p - \mathbf{a}\cdot\mathbf{b} = 0$ then (*) gives $(a\cdot x)\mathbf{b} = \mathbf{c}$ and there are no solutions since $\mathbf{b}$ and $\mathbf{c}$ are map.

If $p = 0$ then the scalar of (*) with $\mathbf{a}$ gives $\mathbf{a}\cdot\mathbf{c} = 0$ so there are solutions only if $\mathbf{a}\cdot\mathbf{c} = 0$, then we have

$$\mathbf{x} = \frac{c - (a\cdot x)\mathbf{b}}{-a\cdot b} \quad \text{if } a\cdot b \neq 0 \quad bfx = -\frac{c}{a\cdot b} + \alpha\mathbf{b} \ \ \alpha\epsilon\mathbf{R}$$

If $p = 0$ and $a\cdot b = 0$ then $(a\cdot x)\mathbf{b} = \mathbf{c}$ then there are no solutions.

We then find $\mathbf{y}$ from $\mathbf{y} = -\mathbf{b}\times\mathbf{x}$