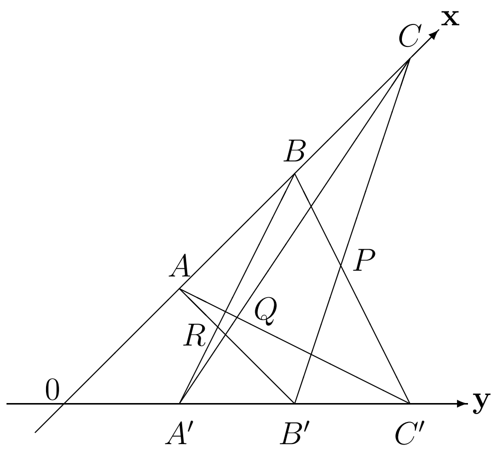
Question
State Pappus' theorem and prove it using vectors.

Answer



Pappus states that
P, Q, R are collinear.

Let the points have position vectors:

$$A : \alpha\mathbf{x} \quad B : \beta\mathbf{x} \quad C : \gamma\mathbf{x} \quad A' : \alpha'\mathbf{y} \quad B' : \beta'\mathbf{y} \quad C' : \gamma'\mathbf{y}$$

Then P: $t = \beta\mathbf{x} - (1-t)\gamma\mathbf{y} = s\gamma\mathbf{x} + (1-s)\beta'\mathbf{y}$
So $t\beta = s\gamma$ and $(1-t)\gamma' = (1-s)\beta'$.
Solving these gives:

$$s = \frac{\beta(\gamma' - \beta')}{\gamma\gamma' - \beta\beta'} \quad 1 - s = \frac{\gamma'(\gamma - \beta)}{\gamma\gamma' - \beta\beta'}$$

So

$$\mathbf{P} : \frac{\beta\gamma(\beta' - \gamma')}{\beta\beta' - \gamma\gamma'}\mathbf{x} + \frac{\beta'\gamma'(\beta - \gamma)}{\beta\beta' - \gamma\gamma'}\mathbf{y}$$

Similarly

$$\mathbf{Q} : \frac{\alpha\gamma(\gamma' - \alpha')}{\gamma\gamma' - \alpha\alpha'}\mathbf{x} + \frac{\gamma'\alpha'(\gamma - \alpha)}{\gamma\gamma' - \alpha\alpha'}\mathbf{y}$$

$$\mathbf{R} : \frac{\alpha\beta(\alpha' - \gamma')}{\alpha\alpha' - \gamma\gamma'}\mathbf{x} + \frac{\alpha'\beta'(\alpha - \beta)}{\alpha\alpha' - \beta\beta'}\mathbf{y}$$

Thus $\alpha\alpha'(\beta\beta' - \gamma\gamma')\mathbf{P} + \beta\beta'(\gamma\gamma' - \alpha\alpha')\mathbf{Q} + \gamma\gamma'(\alpha\alpha' - \beta\beta')\mathbf{R} = \mathbf{0}$
and $\alpha\alpha'(\beta\beta' - \gamma\gamma') + \beta\beta'(\gamma\gamma' - \alpha\alpha') + \gamma\gamma'(\alpha\alpha' - \beta\beta') = 0$
Thus PQR are collinear.