Question

(a) The midpoints of consecutive sides of an arbitrary quadrilateral are joined. Show that the figure so formed is a parallelogram.

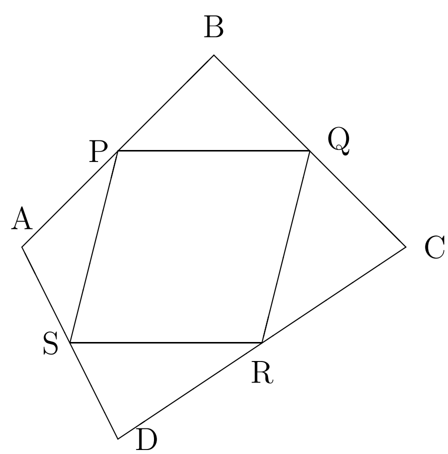
(b) Show that there is a triangle with sides equal and parallel to the medians of any given triangle.

(c) The points D, E, F divide the sides AB, BC, CA, respectively, of the triangle ABC in the ration $m : n$. Show that for any point P in space

$$\vec{PA} + \vec{PB} + \vec{PC} = \vec{PD} + \vec{PE} + \vec{PF}$$

Answer

(a)

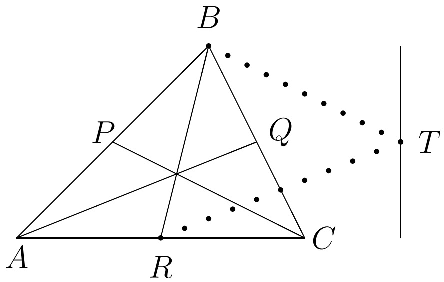


Choose an origin O then:

$$\begin{aligned}
\vec{OP} &= \tfrac{1}{2}(\vec{OA} + \vec{OB}) \\
\vec{OQ} &= \tfrac{1}{2}(\vec{OB} + \vec{OC}) \\
\vec{OR} &= \tfrac{1}{2}(\vec{OC} + \vec{OD}) \\
\vec{OS} &= \tfrac{1}{2}(\vec{OD} + \vec{OA}) \\
\vec{PQ} = \vec{OQ} - \vec{OP} &= \tfrac{1}{2}(\vec{OC} - \vec{OA}) \\
\vec{SR} = \vec{OR} - \vec{OS} &= \tfrac{1}{2}(\vec{OC} - \vec{OA})
\end{aligned}$$

SO $PQ = RS$ and PQ is parallel to RS. So $PQRS$ is a parallelogram.

(b)



Choose B to be the origin, and let $\vec{BA} = \mathbf{a} \quad \vec{BC} = \mathbf{c}$.

Then $\vec{BP} = \frac{1}{2}\mathbf{a} \quad \vec{BQ} = \frac{1}{2}\mathbf{c} \quad \vec{BR} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$

$$\begin{aligned} \vec{AQ} = \vec{BQ} - \vec{BA} &= \tfrac{1}{2}(\mathbf{c} - \mathbf{a}) \\ \vec{CP} = \vec{BP} - \vec{BC} &= \tfrac{1}{2}(\mathbf{a} - \mathbf{c}) \end{aligned}$$

$$\begin{aligned} \text{Let the point } T \text{ be defined by } \vec{BT} &= \tfrac{1}{2}(\vec{BR} + \vec{AQ}) \\ &= \tfrac{1}{2}(\mathbf{a} + \mathbf{c}) + \tfrac{1}{2}\mathbf{c} - \mathbf{a} \\ &= \mathbf{c} - \tfrac{1}{2}\mathbf{a} \end{aligned}$$

Let the point R be defined by $\vec{BR} = \vec{PC} = \mathbf{c} - \frac{1}{2}\mathbf{a}$

So $\vec{BR} = \vec{BT}$ and thus $R = T$

So the triangle BRT is as required, since $\vec{RT} = \vec{BT} - \vec{BR} = \vec{AQ}$ and $\vec{BT} = \vec{PC}$.

Note if the medians form a triangle the sum of the vectors represented then should be zer0, and it is.

(c) By the ration theorem:

$$\begin{aligned} \vec{PD} &= \frac{m\vec{PB} + n\vec{PA}}{m+n} \\ \vec{PE} &= \frac{m\vec{PC} + n\vec{PB}}{m+n} \\ \vec{PF} &= \frac{m\vec{PA} + n\vec{PC}}{m+n} \end{aligned}$$

Adding gives $\vec{PD} + \vec{PE} + \vec{PF} = \vec{PA} + \vec{PB} + \vec{PC}$