QUESTION
Obtain at least one solution of the form

$$y = x^\sigma \sum_{n=0}^{\infty} a_n x^n$$

for each of the following differential equations. Where possible, obtain a second independent solution of the same form, or comment on why it is not possible to do so.
$x^2y'' + xy' + (x^2 - p^2)\, y = 0$
Discuss the cases (a) $2p$ not an integer, (b) $2p$ an integer but p not an integer and (c) p an integer, separately.

ANSWER
$x^2y'' + xy' + (x^2 - p^2)\, y = 0$ This is the Bessel equation. 0 is a regular singular point.
$\sum_{n=0}^{\infty} a_n \{ \left[(\sigma+n)(\sigma+n-1) + (\sigma+n) - p^2 \right] x^{\sigma+n} + x^{\sigma+n+2} \} = 0$
Factorizing
$\sum_{n=0}^{\infty} \{ a_n \left[(\sigma+n)^2 - p^2 \right] x^{\sigma+n} + x^{\sigma+n+2} \} = 0$
Reordering
$a_0 (\sigma^2 - p^2)\, x^\sigma + a_1 \left[(\sigma+1)^2 - p^2 \right] x^{\sigma+1}$
$+ \sum_{n=2}^{\infty} x^{\sigma+n} \{ a_n \left[(\sigma+n)^2 - p^2 \right] + a_{n-2} \} = 0$
$a_0 \neq 0$ gives $\sigma^2 - p^2 = 0 \Rightarrow \sigma = \pm p \Rightarrow a_1 = 0$ (If we assume $a_0 = 0$ and $a_1 \neq 0$, we get the same solution, only written differently. As a convention, we fix σ by assuming $a_0 \neq 0$)
$a_n = -\frac{1}{(n \pm p)^2 - p^2} a_{n-2} = -\frac{1}{n(n \pm 2p)} a_{n-2}$
Discussion:
The difference between the two values of σ is $p - (-p) = 2p$.
(a) $2p$ not an integer: we get two independent Frobenius series solutions.
(b) $2p$ integer but p not an integer: Assume p>0, Then for $\sigma = p$ we obtain a Frobenius solution, but for $\sigma = -p$ we find $a_{2p} = -\frac{x}{1} 2p\,(2p - 2p) a_{2p-2}$ dividing by zero, so this second Frobenius solution does not exist.
(c) p an integer: We get only one series solution, which is now a power series.