QUESTION

Obtain at least one solution of the form

$$y = x^{\sigma} \sum_{n=0}^{\infty} a_n x^n$$

for each of the following differential equations. Where possible, obtain a second independent solution of the same form, or comment on why it is not possible to do so.

$4xy'' + 2y' + y = 0$

ANSWER

$4xy'' + 2y + y = 0$, or $y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0$

y=0 is a regular singular point, so we need a Frobenius series:

$y = \sum_{n=0}^{\infty} a_n x^{\sigma+n}$.

Substitute:

$\sum_{n=0}^{\infty} a_n \left[(\sigma+n)(\sigma+n-1)x^{\sigma+n-2} + \frac{1}{2}(\sigma+n)x^{\sigma+n-2} + \frac{1}{4}x^{\sigma+n-1}\right] = 0$

$\sum_{n=0}^{\infty} a_n \left[(\sigma+n)\left(\sigma+n-\frac{1}{2}\right)x^{\sigma+n-2} + \frac{1}{4}x^{\sigma+n-1}\right] = 0$

Now reorder by powers of x.

Let $n = m+1$ in the first term and $n = m$ in the second.

$\sum_{-1}^{\infty} a_{m+1}(\sigma+m+1)\left(\sigma+m+\frac{1}{2}\right)x^{\sigma+m-1} + \sum_{m=0}^{\infty} \frac{1}{4} a_m x^{\sigma+m-1} = 0$

$m = -1$ gives us $\sigma\left(\sigma - \frac{1}{2}\right) = 0 \Rightarrow \sigma = 0$ or $\sigma = \frac{1}{2}$.

$m = 0$ gives $a_{m+1} = -\frac{1}{4(\sigma+m+1)\left(\sigma+m+\frac{1}{2}\right)} a_m$

Case $\sigma = 0:$ $a_{m+1} = -\frac{1}{(2m+2)(2m+1)} a_m$ The solution is $a_m = \frac{(-1)^m}{(2m)!} a_0$

Case $\sigma = \frac{1}{2}:$ $a_{m+1} = -\frac{1}{(2m+3)(2m+2)} a_m,\ a_m = \frac{(-1)^m}{(2m+1)!} a_0$

The general solution is obtained by adding the two cases:

$$y = A \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n + B \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n+\frac{1}{2}}$$