Question

Let A be the 3×3 matrix $\frac{1}{2}\begin{pmatrix} 1 & -2 & -3 \\ 5 & -26 & -45 \\ -3 & 18 & 31 \end{pmatrix}$. Sketch the dynamics of the linear system $v \mapsto Av$ in $\mathbf{R}$, indicating stable and unstable manifolds.

Answer

The eigenvalues of kA are $k\times$ eigenvalues of A.

Eigenvalues of $\begin{pmatrix} 1 & -2 & -3 \\ 5 & -26 & -45 \\ -3 & 18 & 31 \end{pmatrix}$ are solutions λ to

$(1-\lambda)[(\lambda-31)(\lambda+26)+45.18]+2[5(31-\lambda)-3.45]-3[5.18-3(\lambda+26)]=0$

i.e. $(1-\lambda)[\lambda^2-5\lambda+4]+2[-5\lambda+20]-3[-3\lambda+12]=0$

i.e. $(\lambda-4)[(1-\lambda)(\lambda-1)-10+9]=0$ (do <u>not</u> multiply out the previous line!)

i.e. $(\lambda-4)[-\lambda^2+2\lambda-2]=0$, so $\lambda=4, 1\pm i$.

Hence eigenvalues of the matrix A in the question are $\lambda = \underline{2, \frac{1}{2}(1\pm i)}$.

Therefore the origin is a <u>hyperbolic saddle</u>.

Eigenvectors:

$\lambda = 2:\ \begin{pmatrix} -3 & -2 & -3 \\ 5 & -30 & -45 \\ -3 & 18 & 27 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$ (or any scalar multiple of it).

$$\lambda = \tfrac{1}{2}(1+i):\ \begin{pmatrix} -i & -2 & -3 \\ 5 & -27-i & -45 \\ -3 & 18 & 30-i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{i.e. } \begin{pmatrix} -i & -2 & -3 \\ 0 & -27+9i & -45+15i \\ 0 & 18-6i & 30-10i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which we get $(9-3i)y+(15-5i)z=0$ i.e. $-3y=5z$; then $-ix=\frac{1}{5}y$.

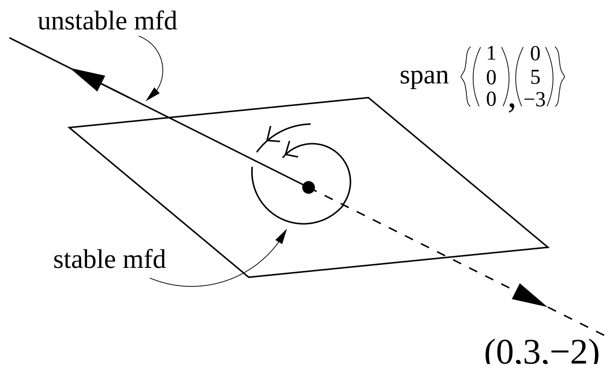
So a (complex) <u>eigenvector</u> for $\lambda=\frac{1}{2}(1+i)$ is $\underline{(i,5,-3)}$.

Hence let $\xi=(0,5,-3),\ \eta=(1,0,0)$

Then $A\xi=\frac{1}{2}\xi-\frac{1}{2}\eta,\ A\eta=\frac{1}{2}\xi+\frac{1}{2}\eta$. Hence in the plane spanned by $\xi,\ \eta$ the action of A is to apply the 2×2 matrix

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}.$$

i.e. <u>rotate</u> by $\dfrac{\pi}{4}$ and <u>shrink</u> by factor $\dfrac{1}{\sqrt{2}}$.


unstable mfd
span $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\5\\-3\end{pmatrix}\right\}$
stable mfd
(0.3,−2)