Question Let A be a real 2×2 matrix.

(i) Show that if the eigenvalues of a are real, and there exist two linearly independent eigenvalues u, v, then the matrix P whose columns are $\{u, v\}$ satisfies $AP = PD$ where D is the 2×2 diagonal matrix whose diagonal entries are the eigenvalues of A.

(ii) Show that if A has repeated real eigenvalue λ and only one independent eigenvector u, then by taking u to be a vector that satisfies $(A - \lambda I)v = u$ and building P from u, v, as above we have $AP = PE$ where E is the 2×2 matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

(iii) Show that if A has a pair of complex eigenvalues $\alpha \pm i\beta$ then, denoting a complex eigenvector corresponding to $\alpha + i\beta$ by $\xi + i\eta$ (ξ, η real vectors), the matrix P whose columns are $\{\xi, \eta\}$ satisfies $AP = P\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.

[The above is the proof that in suitable coordinates every <u>linear</u> system in $\mathbf{R^2}$ takes one of the 3 standard forms.]

Answer

(i) If $Au = \lambda u$, $Av = \mu v$ (maybe $\lambda = \mu$) then the columns of AP are $\{\lambda u, \mu v\}$, so $\underline{AP = PD}$ with $D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

(ii) Here $Au = \lambda u$ and $Av = \lambda v + u$ so we see $\underline{AP = PE}$,
$E = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

(iii) We have $A(\xi + i\zeta) = (\alpha + i\beta)(\xi + i\zeta)$, and a is <u>real</u>, so taking real and imaginary parts we see $A\xi = \alpha\xi - \beta\eta$, $A\eta = \beta\xi + \alpha\eta$. Thus $\underline{AP = P\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$.

In (ii) $u \neq 0$ (by definition of eigenvector), so $v \neq 0$. Indeed, v is not a scalar multiple of u, as $(A - \lambda I)u = 0$ but $(A - \lambda I)v \neq 0$. So $\{u,\ v\}$ are linearly independent. In (iii) neither ξ not $\eta = 0$ (else $A(\xi + i\zeta) = (\alpha + i\beta)(\xi + i\zeta)$ gives $\beta = 0$), and if $\xi = k\zeta, k \in \mathbf{R}$, then $(\alpha k - \beta)\zeta = A\xi = kA\zeta = k(k\beta + \alpha)\zeta$, giving $k^2 = -1$: contradiction. So in all cases P is invertible, so $P^{-1}AP = D, E$ or $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.