Question
A two-dimensional viscous unsteady flow takes place in the (x, y)-plabe. Ther are no body forces. The fluid is incompressible and hsas constant density ρ and constant dynamic viscosity μ. Show that, when the Reynolds number Re is much less than one, the stream function $\phi(x, y)$ satisfies the biharmonic equation

$$\nabla^4 \phi = 0.$$

Explain briefly why, even for unsteady flwo, there are no time derivatives in this equation and comment on how temporal changes would enter the problem for a truly unsteady flow.
Steady low Reynolds number two-dimensional flow takes place in a wedge of semi-angle α, the flow beinh driven by a shearing mechanism far away from the corner of the wege. Give the boundary condtions that must be satisfied by the stream function ϕ and its derivatives on the wedge walls $\theta = \pm\alpha$, where $(r, \theta 00$ are plane polar coordinates. Verify that a flow with stream function

$$\phi(r, \theta) = r^\lambda (A \cos \lambda\theta + B \cos(\lambda - 2)\theta)$$

is possible for non-zero A and B only if the constant λ satisfies

$$(\lambda - 2) \tan((\lambda - 2)\alpha) = \lambda \tan \lambda\alpha.$$

[If you wish you may use, without proof, the fact that in cylindrical polar coordinates (r, θ, z)

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.]$$

Answer
We have 2/D unsteady flow in the (x, y)-plane, so as usual the continuity equation is automatically satisfied by $\psi = \psi(x, y, t)$ where $u = \psi_y$, $v = -\psi_x$. The Navier-Stokes momentum equations are

$$\underline{q}_t + (\underline{q}.\nabla)\underline{q} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{q}$$

Non-dimensionalising using

$$\begin{aligned} \underline{q} &= U_\infty \underline{q}' \\ \underline{x} &= L\underline{x}' \\ p &= \mu\frac{U_\infty}{l}p' \end{aligned}$$

where U_∞, L are a typical velocity and length in the flow, we find that (with $t = (L/U_\infty t'))$
(dropping primes)

$$\frac{U_\infty^2}{L}(\underline{q}_t + (\underline{q}.\nabla)\underline{q}) = -\frac{\mu U_\infty}{\rho L^2}\nabla^2\underline{q}$$

$\Rightarrow Re(\underline{q}_t + (\underline{q}.\nabla)\underline{q}) = -\nabla p + \nabla^2\underline{q} \quad (Re = LU_\infty/\nu)$.
So for $Re \ll 1$ we get, to lowest order,

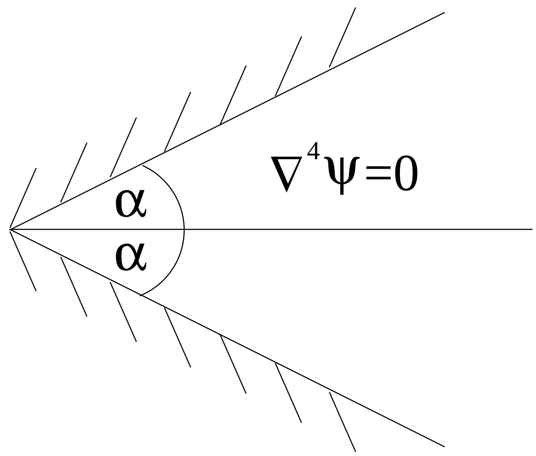
$$\nabla p = \nabla^2\underline{q}.$$

$\Rightarrow curl\nabla p = curl\nabla^2\underline{q} = \nabla^2 curl\underline{q} = 0$

Now $curl\underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \psi_x & -\psi_y & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\psi_{xx} - \psi_{yy} \end{pmatrix}$

$\Rightarrow$ for slow flow $\nabla^4\psi = 0$

This equation contains no time derivatives as all inertia has vanished to lowest order. In a truly unsteady flow the time derivative would manifest itself via the boundary conditions.



Now we must solve $\nabla^4\psi = 0$ $(r \geq 0,\ -\alpha \leq \theta \leq \alpha)$.
Suitable B/C's are no slip:-

$$\psi = \psi_\theta = 0 \quad \text{on } \theta = \pm\alpha$$

[also symmetry conditions could be used]
So try $\psi = r^\lambda(A\cos\lambda\theta + B\cos(\lambda-2)\theta)$

$$\begin{aligned}
\nabla^2\psi &= \lambda(\lambda-1)r^{\lambda-2}(A\cos\lambda\theta + B\cos(\lambda-2)\theta \\
&\quad +\tfrac{1}{r}\lambda r^{\lambda-1}(A\cos\lambda\theta + B\cos(\lambda-2)\theta) \\
&\quad +\frac{r^\lambda}{r^2}(-\lambda^2 A\cos\lambda\theta - (\lambda-2)^2 B\cos(\lambda-2)\theta) \\
&= (\lambda(\lambda-1)+\lambda)r^{\lambda-2}(A\cos\lambda\theta - B\cos(\lambda-2)\theta) \\
&\quad +r^{\lambda-2}(-\lambda^2 A\cos\lambda\theta - (\lambda-2)^2 B\cos(\lambda-2)\theta) \\
&= B(\lambda^2 - (\lambda-2)^2)\cos(\lambda-2)\theta(r^{\lambda-2})
\end{aligned}$$

$$\begin{aligned}
\text{so } \nabla^4\psi &\propto (\lambda-2)(\lambda-3)r^{\lambda-4}\cos(\lambda-2)\theta + (\lambda-2)r^{\lambda-4}\cos(\lambda-2) \\
&\quad -r^{\lambda-4}(\lambda-2)^2\cos(\lambda-2)\theta \\
&= (\lambda-2)(\lambda-3+1-\lambda+2) \\
&= 0
\end{aligned}$$

Thus

$$\nabla^4\psi = 0$$

B/C's:- (ψ is even, so need only impose at $\theta = +\alpha$)

$\psi = 0$ at $\theta = \alpha$:- $\quad 0 = A\cos\lambda\alpha + B\cos(\lambda-2)\alpha$

$\psi_\theta = 0$ at $\theta = \alpha$:- $\quad 0 = A\lambda\sin\lambda\alpha + (\lambda-2)B\sin(\lambda-2)\alpha$

These homgeneous equations have no non-zero solution unless the determinant of the coefficients is zero.

i.e. need $\begin{vmatrix} \cos\lambda\alpha & \cos(\lambda-2)\alpha \\ \lambda\sin\lambda\alpha & (\lambda-2)\sin(\lambda-2)\alpha \end{vmatrix} = 0$

$$\Rightarrow (\cos \lambda\alpha)(\lambda - 2)\sin(\lambda - 2) - \cos(\lambda - 2)\alpha(\lambda \sin \alpha) = 0$$

$$\Rightarrow (\lambda - 2)\tan(\lambda - 2)\alpha = \lambda \tan \lambda\alpha$$