Question

An axisymmetric jet of viscous incompressible fluid of consatn density ρ and constant kinematice viscosity ν is injected in the positive z-direction through a jole at $r = z = 0$ into the same fluid at rest. Cylindrical polar coordinates (r, θ, z) are used and the flow is independent of θ so that the fluid velocity is given by $\underline{w} = u\underline{e}_r + w\underline{e}_z$ where $\underline{e}_r$ and $\underline{e}_z$ are unit vectors in the $r-$ and $z-$ directions respectively. YOU MAY ASSUME that in a boundary layer treatment of the flow where the radius of the jet is assumed to be much less than its length, the (dimensional) boundary layer equations are

$$\begin{aligned} uw_r + ww_z &= \nu\left(w_{rr} + \frac{1}{r}w_r\right) \\ (ru)_r + (rw)_z &= 0 \end{aligned}$$

(i) Assuming that bot ruw and rw_r then to zero as $r \to \infty$, show that the quantity

$$M = 2\pi\rho\int_0^\infty rw^2\,dr$$

is independent of z and give a physical interpretation of this result. State two condtions that must be satisfies by the velocity on the z-axis and give brief reasons why these must hold.

(ii) By using the fact that $dM/dz = 0$ to help determine m and n, verify that a similarity solution exists to the problem in the form

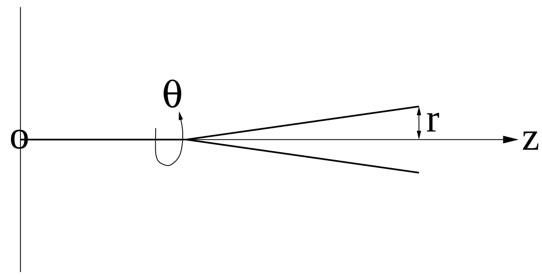
$$\begin{aligned} \psi &= z^m f(\mu) \\ \mu &= rz^{-n} \\ (ru &= -\phi_z, \quad rw = \phi_r) \end{aligned}$$

provided f satisfies the ordinary differential equation

$$ff' - \mu(ff'' + f'^2) = \nu(f' - \mu f'' + \mu^2 f''')$$

and give suitable boundary conditions for this equation.

Answer



$\underline{q} = u\underline{\hat{e}}_r + w\underline{\hat{e}}_z$

We may assume that

$$\begin{aligned} uw_r + ww_z &= \nu(w_{rr} + \frac{1}{r}w_r) \\ (ru)_r + (rw)_z &= 0 \end{aligned}$$

(i) Consider

$$M = 2\pi\rho \int_0^\infty rw^2\,dr, \quad \frac{dM}{dz} = 2\pi\rho \int_0^\infty 2rww_z\,dr$$

Using momentum equation

$\Rightarrow \dfrac{dM}{dz} = 2\pi\rho \displaystyle\int_0^\infty 2r(\nu(w_{rr} = \frac{1}{r}w_r) - uw_r)\,dr$

Integrate by parts:-

$$\begin{aligned} \frac{dM}{dz} &= 4\pi\rho \left\{ \int_0^\infty \nu rw_{rr} + \nu w_r - ruw_r\,dr \right\} \\ &= 4\pi\rho \left\{ [\nu rw_r - ru_w]_0^\infty - \int_0^\infty \nu w_r - \nu w_r - (ru)_r w\,dr \right\} \end{aligned}$$

Now since $rw_r,\ ruw$ are zero at 0 by symmetry and $\to 0$ as $r \to \infty$ we have

$$\begin{aligned} \frac{dM}{dz} &= 4\pi\rho \int_0^\infty (ru)_r w_r\,dr \\ &= -4\pi\rho \int_0^\infty (rw)_z w \\ &= -4\pi\rho \int_0^\infty rww_z\,dr \end{aligned}$$

Thus $\frac{dM}{dz} = -\frac{dM}{dz}$ so $\frac{dM}{dz} = 0$ and M is independent of z.

Physically the result means that the momentum flux of the jet in conserved along the jet (i.e. is the same $\forall z$).

By symmetry we require $u = w_r = 0$ at $r = 0$

(ii) With $ru = -\psi_z,\ rw = \psi_r$ the continuity equation is automatically satisfied.

$$\begin{aligned}
\text{With } \psi &= z^m f(\eta),\ \eta = rz^{-n} \text{ we have} \\
u &= -\psi_z/r = -mr^{-1}z^{m-1}f + nz^{m-n-1}f' \\
w &= \psi_r/r = r^{-1}z^{m-n}f' \\
w_z &= r^{-1}(m-n)z^{m-n-1}f' - nz6m - 2n - 1f'' \\
w_r &= -r^{-2}z^{m-n}f' + r^{-1}z^{m-2n}f'' \\
w_{rr} &= 2r^{-3}z^{m-n}f' - 2r^{-2}z^{m-2n}f'' + r^{-1}z^{m-3n}f'''
\end{aligned}$$

$$\Rightarrow (-mr^{-1}z^{m-1}f + nz^{m-n-1}f')(-r^{-2}z^{m-n}f' + r^{-1}z^{m-2n}f'')$$

$$+r^{-1}z^{m-n}f'(r^{-1}(m-n)z^{m-n-1}f' - nz^{m-n}f'')$$

$$= \nu(2r^{-3}z^{m-n}f' - 2r^{-2}z^{m-2n}f'' + r^{-1}z^{m-3n}f''' + r^{-2}z^{m-2n}f''$$

$$-r^{-3}z^{m-n}f')$$

At this stage consider the fact that M is independent of z.

Thus $\displaystyle\int_0^\infty \frac{r}{r^2} z^{2m-2n}(f'(\eta))^2 \frac{dr}{d\eta}\, d\eta$ must not depend upon z.

$$\begin{aligned}
\Rightarrow \int_0^\infty r^{-1}\frac{z^{2m-2n}}{z^{-n}}(f'(\eta))^2\, d\eta &= \int_0^\infty \frac{z^{2m-2n}}{\eta}(f'(\eta))^2\, d\eta \\
&= 0 \\
&\Rightarrow m = n
\end{aligned}$$

Thus

$$\begin{gathered}
(-mr^{-1}z^{m-1}f + mz^{-1}f')(-r^{-2}f' + r^{-1}z^{-m}f'') \\
+r^{-1}f'(-mz^{-m-1}f'') \\
= \mu(r^{-3}f' - r^{-2}z^{-m}f'' + r^{-1}z^{-2m}f''')
\end{gathered}$$

Thence

$$\begin{aligned}
& mr^{-3}z^{m-1}ff' - mr^{-2}z^{-1}ff'' - mr^{-2}z^{-1}f'^2 \\
=\ & \nu(r^{-3}f' - r^{-2}z^{-m}f'' + r^{-1}z^{-2m}f''') \\
& mz^{m-1}ff' - mrz^{-1}ff'' - mrz^{-1}f'^2 \\
=\ & \nu(f' - rz^{-m}f'' + r^2z^{-2m}f''')
\end{aligned}$$

Comparing the first terms on each side $\Rightarrow m = 1$

$$
\begin{aligned}
\Rightarrow \qquad & ff' - (r/z)ff'' - (r/z)f'^2 \\
& = \nu(f' - (r/z)f'' + (r^2/z^2)f'') \\
& ff' - \eta ff'' - \eta f'^2 \; = \; \nu(f' - \eta f'' + \eta^2 f''') \\
i.e. \qquad & ff' - \eta(ff'' + f'^2) \; = \; \nu(f' - \eta f'' + \eta^2 f''')
\end{aligned}
$$

B/C's:- $u = w_r = 0$ at $r = 0 \Rightarrow \quad f(0) = f'(0) = 0$

Flux condition $\Rightarrow \int_0^\infty \frac{(f'(\eta))^2}{\eta} \, d\eta =$ given constant.