Question

Steady viscous incompessible flow takes place parallel to the z-axis in a cylindrical pip of cross section D. The kinematic viscosity and density of the fluid are both constant and are given by ν and ρ respectively and there are no body forces. By assuming a velocity of the form $\underline{q} = (0, 0, w(x, y))$, show that the pressure gradient $-P$ in the z-direction must be constant. Show also that w satisfies the equation

$$\nabla^2 w = A,$$

where A is a constant that you should specify, and give suitable boundary conditions for w.

When D is a circular disc of radius a, find w and show that the mass flus M is given by the Hagen-Poiseulle discharge formula.

$$M = \frac{\pi P a^4}{8\nu}.$$

Suppose now that viscous incompressible flow takes place in the same circular pipe but is driven by an *unsteady* pressure gradient $p_z = p_0 e^{-k^2 t}$ where t denores time and p_0 and k are constants. By seeking a solution of the form $\underline{q} = (0, 0, R(r)e^{-k^2 t})$, show that the function R satisfies the ordinary differential equation

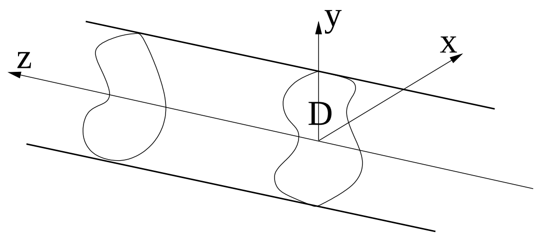
$$R'' + \frac{1}{r}R' + AR = B$$

where A and B are constants which you should specify. Give suitable boundary conditions for this equation.

[You may use, without proof, the fact that in cylindrical polar coordinates (r, θ, z)

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.]$$

Answer



Assume $\underline{q} = (0, 0, w(x, y))$ then $div(\underline{q}) = 0$
The three components of the Navier-Stokes equations become

$$\left.\begin{aligned} 0 &= -p_x/\rho + 0 \\ 0 &= -p_y/\rho + 0 \\ 0 &= -p_z/\rho + \nu(w_{xx} + w_{yy}) \end{aligned}\right\} \Rightarrow p = p(z) \text{ only.}$$

Now by the standard separation of variables argument, since $w = w(x, y)$ only we have

$$\begin{aligned} -p_z/rho &= constant \\ \Rightarrow \quad -p_z/\rho &= P/\rho \text{ say. (P constant)} \\ \Rightarrow \quad 0 &= P/\rho + \nu \nabla^2 w \\ \Rightarrow \quad \nabla^2 w &= -P/mu \quad (\underline{x} \in D) \end{aligned}$$

Boundary conditions:- by no-slip we need $w = 0$ on δD.
If D is a circle of radius a, then the problem becomes (assuming symmetry so that $w = w(r)$ only)

$$\begin{aligned} \frac{1}{r}(rw_r)_r &= -\frac{p}{\mu} \quad (w = 0 \text{ on r = a}) \\ \Rightarrow \quad (rw_r)_r &= -\frac{Pr}{\mu}, \\ rw_r &= -\frac{Pr^2}{2\mu} + C \\ \Rightarrow \quad w_r &= -\frac{Pr}{2\mu} + \frac{C}{r} \end{aligned}$$

So $w = -\dfrac{Pr^2}{4\mu} + C \log r + D$.
Must choose $C = 0$ so that $w < \infty$ at $r = 0$
Now imposing $w(a) = 1 \quad \Rightarrow$

$$w = \frac{P}{4\mu}(a^2 - r^2)$$

$$\begin{aligned} M &= \int_{pipe} \rho w \, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{\rho P}{4\mu}(a^2 - r^2)\, r dr d\theta \\ &= 2\pi \left[\frac{\rho P}{4\mu} \left(\frac{a^2 r^2}{2} - \frac{r^4}{4} \right) \right]_0^a = \frac{2\pi \rho P}{4\mu} \frac{a^4}{4} \end{aligned}$$

Thus as required

$$M = \frac{\pi P a^4}{8\nu}$$

Now the flow is UNSTEADY with $p_z = -p_o e^{-k^2 t}$.
Again we seek $\underline{q} = (0, 0, w)$ where now w is a function of r and t. So still $div(\underline{q}) = 0$.
The only N/S equation that does not give $0 = 0$ is

$$w_t = -p_x/\rho + \nu \nabla^2 W$$

So, with w as suggested, we find that

$$\begin{aligned} -k^2 R e^{k^2 t} &= -\frac{p_0}{\rho} e^{-k^2 t} + \nu e^{-k^2 t} \frac{1}{r}(rR')' \\ \Rightarrow \qquad -k^2 R &= -\frac{p_0}{\rho} + \frac{\nu}{r}(R' + rR'') \\ \Rightarrow \quad \nu R'' + \frac{\nu R'}{r} + k^2 R &= \frac{p_0}{\rho} \\ i.e. \qquad R'' + \frac{R'}{r} + AR &= B \end{aligned}$$

$$(A = \frac{k^2}{\nu}, \quad B = \frac{p_0}{\nu \rho})$$

Boundary conditions:- need $R(a) = 0$