QUESTION

(More difficult) Let $f(z) = \begin{cases} z^5/|z^4|, & z \neq 0, \\ 0, & z = 0 \end{cases}$

and let $g(z) = \frac{f(z)}{z}$. Find $g(z)$ for z on the real axis and also for z on the line $y = x$. Deduce that f is is not differentiable at $z = 0$. Now writing $f = u + iv$, where u and v are real, find $u(x,0), v(x,0), u(0,y), v(0,y)$ and show that the Cauchy-Riemann equations hold at $z = 0$. Comment.

ANSWER

If $z \neq 0$, $g(z) = \frac{f(z)}{z} = \frac{z^4}{|z^4|}$. Hence on the real axis $g(z) = 1$. On the line $y = x$, $z = |z|e^{i\pi/4}$ so $\frac{z^4}{|z^4|} = e^{i\pi} = -1$

Now $f'(z) = lim_{h\to 0}\frac{f(h)}{h} = lim_{h\to 0} g(h)$ which we have shown does not exist as it depends on the direction that we approach 0. Thus $f(z)$ is not differentiable at $z = 0$. Now let $f(z) = u(x,y) + iv(x,y) = (x+iy)^5/(x^4+y^4)$. Putting $y = 0$, we get $u(x,0) + iv(x,0) = x$, so $u(x,0) = x$, $v(x,0) = 0$. Similarly, $u(0,y) = 0$, $v(0,y) = y$. Thus $u_x = v_y = 1$ and $u_y = -v_x = 0$, and the Cauchy-Riemann equations hold. There is no contradiction as the Cauchy-Riemann equations do not *imply* differentiability.