QUESTION

Consider the following modification of the classic Buffon's needle experiment: two needles of the same length, ℓ, fused at right angles at their centres, are thrown on a horizontal ruled floor, with parallel lines at a distance $d = \ell$ apart. Let X and Y be binary variables, indicating whether each of the needles crosses a line. Let $Z = X + Y$ be the number of times that this cross arrangement intercepts a line.

(i) Find Prob(Z = 0), Prob(Z = 1) and Prob(Z = 2).

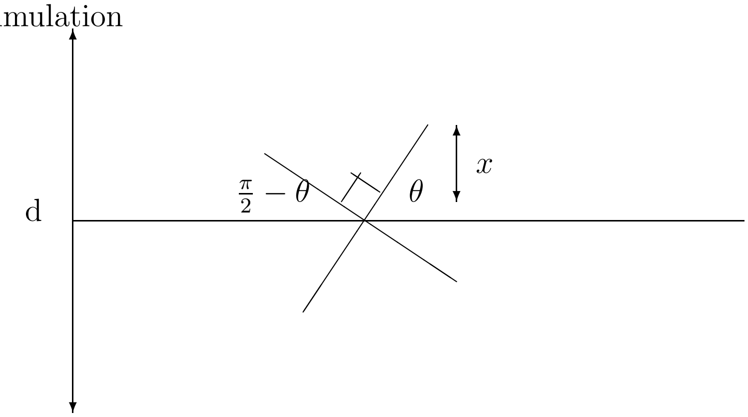
(ii) Find the variance of Z.

(iii) Using the fact that var(Z) =var(X)+var(Y)+2cov(X, Y), find cov(X, Y).

(iv) Comment on the use of this cross arrangement as a variance reduction technique.

ANSWER

Simulation



(i) $P(Z = 2) = (A + B)/(\pi l/2)$

$P(Z = 0) = (C + D)/(\pi l/2)$

$P(Z = 1) = 1 - P(Z = 0) - P(Z = 2)$

$$\begin{aligned} A + B &= 2 \times \left\{ \int_0^{\frac{\pi}{4}} \frac{l}{2} \sin\theta \, d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{l}{2} \sin\left(\frac{\pi}{2} - \theta\right) \, d\theta \right\} \\ &= l \times \Big[-\cos\theta \Big]_0^{\frac{\pi}{4}} + l \times \left[\cos\left(\frac{\pi}{2} - \theta\right) \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= l \times \left[-\frac{\sqrt{2}}{2} + 1 + 1 - \frac{\sqrt{2}}{2} \right] = l \left(2 - \sqrt{2}\right) \end{aligned}$$

$$P(Z=2) = \left[l(2-\sqrt{2})\right]/(\pi l/2) = 2(2-\sqrt{2})/\pi$$

$$\begin{aligned} C+D &= 2 \times \left\{ \frac{\pi l}{4} - \int_0^{\frac{\pi}{4}} \frac{l}{2} \sin\left(\frac{\pi}{2}-\theta\right) d\theta - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{l}{2} \sin\theta \, d\theta \right\} \\ &= l \times \left[\frac{\pi}{2} - \cos\left(\frac{\pi}{2}-\theta\right)\right]_0^{\frac{\pi}{4}} + l \times \Big[\cos\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= l \times \left(\frac{\pi}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = l \times \left(\frac{\pi}{2} - \sqrt{2}\right) \end{aligned}$$

$$P(Z=0) = \left[l\left(\frac{\pi}{2}-\sqrt{2}\right)\right]/(\pi l/2) = \left(\pi - 2\sqrt{2}\right)/\pi$$

$$P(Z=1) = 1 - \frac{2(2-\sqrt{2})}{\pi} - \frac{(\pi-2\sqrt{2})}{\pi} = \frac{4(\sqrt{2}-1)}{\pi}$$

Hence $P(Z=0) = 0.0997, \quad P(Z=1) = 0.527, \quad P(Z=2) = 0.373$

(ii) $E(Z) = 1.273, \quad E(Z^2) = 2.019, \quad \text{var}(Z) = 0.398$

(iii) X and Y have the same distribution:

$$\begin{aligned} P(X=1) &= \frac{2}{\pi l/2} \int_0^{\frac{\pi}{2}} \frac{l}{2} \sin\theta \, d\theta \\ &= \frac{2}{\pi}\Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} = \frac{2}{\pi} \\ P(X=0) &= 1 - \frac{2}{\pi} \end{aligned}$$

$$E(X) = \frac{2}{\pi}, \quad \text{var}(X) = \frac{2}{\pi}\left(1-\frac{2}{\pi}\right) = \text{var}(Y)$$

Therefore $\text{cov}(X,Y) = (0.398 - 2 \times 0.231)/2 = -0.032$

(iv) Observe that $\text{var}\left(\frac{Z}{2}\right) = \frac{1}{4}\text{var}(Z) = 0.0995$

If we throw one needle twice on the floor,

$$\text{var}\left(\frac{X_1+X_2}{2}\right) = \frac{\text{var}(X)}{2} = 0.1155$$

The size of the cross arrangement therefore not only speeds up the process but also reduces the variance