QUESTION

Assume that admissions to an Intensive Care Unit (ICU) are either "emergencies" or "planned". Both the emergency and the planned patients arrive independently at the hospital, according to a Poisson process with rates 1.23 and 0.37 per day, respectively. Their length of stay in the ICU follows a negative exponential distribution with mean 4.17 days. The ICU has 6 beds, and planned patients are only admitted if there are two or more beds available. Emergency patients are only turned away if no beds are available. Patients, planned or emergencies, who are not admitted to the unit are transferred out of the system to another hospital.

(i) Draw the rate diagram for this queuing system.

(ii) Find the average number of beds occupied.

(iii) Find the proportion of emergency patients that have to be transferred out.

(iv) Find the proportion of planned patients that have to be transferred out.

ANSWER

Queues

(i)

$$\begin{tikzcd}
0 \arrow[r, bend left, "(\lambda_e+\lambda_p)\pi_0"] & 1 \arrow[l, bend left, "\mu\pi_1"] \arrow[r, bend left, "(\lambda_e+\lambda_p)\pi_1"] & 2 \arrow[l, bend left, "2\mu\pi_2"] \arrow[r, bend left, "(\lambda_e+\lambda_p)\pi_2"] & 3 \arrow[l, bend left, "3\mu\pi_3"] \arrow[r, bend left, "(\lambda_e+\lambda_p)\pi_3"] & 4 \arrow[l, bend left, "4\mu\pi_4"] \arrow[r, bend left, "(\lambda_e+\lambda_p)\pi_4"] & 5 \arrow[l, bend left, "5\mu\pi_5"] \arrow[r, bend left, "\lambda_e\pi_5"] & 6 \arrow[l, bend left, "6\mu\pi_6"]
\end{tikzcd}$$

[including explanation for variables]

(ii)

$$\begin{aligned}
(\lambda_e+\lambda_p)\pi_0 &= \mu\pi_1 \\
(\lambda_e+\lambda_p)\pi_1 + \mu\pi_1 &= (\lambda_e+\lambda_p)\pi_0 + 2\mu\pi_2 \\
(\lambda_e+\lambda_p)\pi_2 + 2\mu\pi_2 &= (\lambda_e+\lambda_p)\pi_1 + 3\mu\pi_3 \\
(\lambda_e+\lambda_p)\pi_3 + 3\mu\pi_3 &= (\lambda_e+\lambda_p)\pi_2 + 4\mu\pi_4 \\
(\lambda_e+\lambda_p)\pi_4 + 4\mu\pi_4 &= (\lambda_e+\lambda_p)\pi_3 + 5\mu\pi_5 \\
\lambda_e\pi_5 + 5\mu\pi_5 &= (\lambda_e+\lambda_p)\pi_4 + 6\mu\pi_6
\end{aligned}$$

$$\sum_{i=0}^{6} \pi_i = 1$$

Hence

$$\begin{aligned}
(\lambda_e + \lambda_p)\pi_0 &= \mu\pi_1 \\
(\lambda_e + \lambda_p)\pi_1 &= 2\mu\pi_2 \\
(\lambda_e + \lambda_p)\pi_2 &= 3\mu\pi_3 \\
(\lambda_e + \lambda_p)\pi_3 &= 4\mu\pi_4 \\
(\lambda_e + \lambda_p)\pi_4 &= 5\mu\pi_5 \\
\lambda_e\pi_5 &= 6\mu\pi_6
\end{aligned}$$

and therefore,

$$\begin{aligned}
\pi_1 &= \frac{(\lambda_e + \lambda_p)}{\mu}\pi_0 \\
\pi_2 &= \frac{(\lambda_e + \lambda_p)}{2\mu}\pi_1 \\
\pi_3 &= \frac{(\lambda_e + \lambda_p)}{3\mu}\pi_2 \\
\pi_4 &= \frac{(\lambda_e + \lambda_p)}{4\mu}\pi_3 \\
\pi_5 &= \frac{(\lambda_e + \lambda_p)}{5\mu}\pi_4 \\
\pi_6 &= \frac{\lambda_e}{6\mu}\pi_5
\end{aligned}$$

As $\lambda_e = 1.23/\text{day}$, $\lambda_p = 0.37/\text{day}$ and $\mu = \frac{1}{4.17} = 0.24/\text{day}$ we have

$$\pi_i = \frac{1.6^i}{i!0.24^i}\pi_0 = \left(\frac{1.6}{2.4}\right)^i \frac{\pi_0}{i!};\ \ i = 1 \ldots 5$$

$$\pi_6 = \frac{1.23 \times 1.6^5}{6!0.24^6}\pi_0$$

Using $\sum_0^6 \pi_i = 1$ we have

$$
\begin{aligned}
\pi_0 &= 0.00274 \\
\pi_1 &= 0.018 \\
\pi_2 &= 0.061 \\
\pi_3 &= 0.135 \\
\pi_4 &= 0.225 \\
\pi_5 &= 0.301 \\
\pi)6 &= 0.257
\end{aligned}
$$

The average number of beds occupied is

$$0.\pi_0 + 1.\pi_1 + 2.\pi_2 + 3.\pi_3 + \ldots + 6.\pi_6 = 4.49$$

(iii) $\pi_6 = 25.7\%$

(iv) $\pi_5 + \pi_6 = 55.7\%$