QUESTION

(a) Use the simplex method to find *all* optimal solutions of the following linear programming problem.

$$\begin{array}{ll} \text{Maximize} & z = -2x_1 + 7x_2 + 4x_3 \\ \text{subject to} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \\ & 2x_1 - 5x_2 \leq 11 \\ & -x_1 + 3x_2 + x_3 = 7 \\ & x_1 - 8x_2 + 4x_3 \geq 33. \end{array}$$

(b) A company needs to lease warehouse space over the next five months. The requirements for space in thousands of square feet are shown in the following table.

	Month 1	Month 2	Month 3	Month 4	Month 5
Required space	45	30	50	20	60

It is possible to lease the exact space needed on a month-by-month basis. However, the cost per month becomes less as the period of lease increases, so it may be more economical to lease the maximum space needed for the entire five-month period. A variety of intermediate strategies are also possible which may involve starting two or more leases, each having a different leasing period, in some months. The cost per month of different length leases is shown in the following table.

Leasing period (months)	1	2	3	4	5
Cost (£) per 1000 square feet per month	600	480	420	380	350

Write down a linear programming formulation (but do not attempt to solve it) for the problem of finding a leasing policy so that the space requirements for the next five months are met at minimum total leasing cost.

ANSWER

(a) Add slack variables s_1, s_2 and artificial variables a_1, a_2.

Basic	z'	z	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	0	0	2	-5	0	1	0	0	0	11
a_1	0	0	-1	3	1	0	0	1	1	7
a_2	0	0	1	-8	4	0	-1	0	1	33
	1							$+1$	$+1$	0
	1	0	0	5	-5	0	1	0	0	-40
	0	1	2	-7	-4	0	0	0	0	0

Basic	z'	z	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	0	0	2	-5	0	1	0	0	0	11
x_3	0	0	-1	3	1	0	0	1	1	7
a_2	0	0	5	-20	0	0	-1	-4	1	5
	1	0	-5	20	0	0	1	5	0	-5
	0	1	-2	5	0	0	0	4	0	28

Basic	z'	$-z$	$-x_1$	$-x_2$	x_3	s_1	s_2	a_1	a_2	
s_1	0	0	0	3	0	1	$\frac{2}{5}$	$\frac{8}{5}$	$-\frac{2}{5}$	9
x_3	0	0	0	-1	1	0	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1
	-1	0	0	0	0	0	0	1	1	0
	0	1	0	-3	0	0	$-\frac{2}{5}$	$\frac{12}{5}$	$\frac{2}{5}$	30

Basic	x	x_1	x_2	x_3	s_1	s_2	
x_2	0	0	1	0	$\frac{1}{3}$	$\frac{2}{15}$	3
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{15}$	11
x_1	0	1	0	0	$\frac{4}{3}$	$\frac{1}{3}$	13
	1	0	0	0	1	0	39

Solution is $x_1 = 13,\ x_2 = 3,\ x_3 = 11,\ z = 39$

Perform another iteration to find an alternative optimal solution.

Basic	x	x_1	x_2	x_3	s_1	s_2	
s_2	0	0	$\frac{15}{2}$	0	$\frac{5}{2}$	1	$\frac{45}{2}$
x_3	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{11}{2}$
	1	0	0	0	1	0	39

An alternative optimal solution is $x_1 = \frac{11}{2},\ x_2 = 0,\ x_3 = \frac{25}{2},\ z = 39.$

Thus $(x_1, x_2, x_3) = \alpha(13, 3, 11) + (1-\alpha)\left(\frac{11}{2}, 0, \frac{25}{2}\right)$ for $0 \le \alpha \le 1$ is the class of all optimal solutions.

(b) Let $x_{11},\ x_{12},\ x_{13},\ x_{14},\ x_{15}$ be the number of square feet leased at the start of month 1 for 1, 2, 3, 4, 5 months respectively.

Variables

$x_{21},\ x_{22},\ x_{23},\ x_{24}$	for month 2
$x_{31},\ x_{32},\ x_{33}$	for month 3
$x_{41},\ x_{42}$	for month 4
x_{51}	for month 5

are defined similarly.

Maximize

$$z = 600(x_{11} + x_{21} + x_{31} + x_{41} + x_{51})$$

$$
\begin{aligned}
&+960(x_{12}+x_{22}+x_{32}+x_{42})\\
&+1260(x_{13}+x_{23}+x_{33})\\
&+1520(x_{14}+x_{24})+1750x_{15}
\end{aligned}
$$

subject to $x_{ij} \geq 0$ all i, j.

$$
\begin{aligned}
x_{11}+x_{12}+x_{13}+x_{14}+x_{15} &\geq 45\\
x_{12}+x_{13}+x_{14}+x_{15}&\\
+x_{21}+x_{22}+x_{23}+x_{24} &\geq 30\\
x_{13}+x_{14}+x_{15}&\\
+x_{22}+x_{23}+x_{24}&\\
+x_{31}+x_{32}+x_{33} &\geq 50\\
x_{14}+x_{15}+x_{23}+x_{24}&\\
+x_{32}+x_{33}+x_{41}+x_{42} &\geq 20\\
x_{15}+x_{24}+x_{33}+x_{42}+x_{51} &\geq 60
\end{aligned}
$$