Question

Show that any non-empty open set in $\mathbf{R^2}$ (or $\mathbf{R^n}$) can be expressed as the union of a countable collection of closed rectangles.

Answer

Consider the collection $\mathcal{C}$ of all rectangles $R = \{\mathbf{x} | a_r \leq x_r \leq b_r\}$ contained in S, an open set, where a_r and b_r are rational.
Then $\mathcal{C}$ is countable and $\bigcup_{\mathcal{C}} R \subseteq S$.
If $x \epsilon S$ then there is a neighbourhood $N_\epsilon(x) \subseteq S$
$N_\epsilon(x)$ contains a member of $\mathcal{C}$ containing x.
So $x \epsilon \bigcup_{\mathcal{C}} R$, therefore $S = \bigcup_{\mathcal{C}} R$