Question

Let $\{S_m\}$ be a sequence of sets.

Let $H_n = S_n \cap \left(\bigcup_{m<n} S_m \right)^C$

Show that $H_m \cap H_n = \phi \quad$ for $m \neq n$, and that $\bigcup_{n=1}^{\infty} H_n = \bigcup_{n=1}^{\infty} S_n$

Answer

Suppose without loss of generality $m < n$

$H_n = S_n \cap \bigcap_{t<n} (S_t)^C \subseteq (S_m)^C$

$H_m = S_m \cap (\bigcup_{t<m} (S_t))^C \subseteq S_m$

Hence $H_n \cap H_m = \phi$

Since $H_n \subseteq S_n, \quad \bigcup_{n=1}^{\infty} H_n \subseteq \bigcup_{n=1}^{\infty} S_n$

Now suppose $x \epsilon \bigcup_{n=1}^{\infty} S_n$. Let r be the smallest integer such that $x \epsilon S_r$, then $x \not\epsilon S_m$, for $m < r$.

Therefore $x \epsilon S_r \cap (\bigcup_{m<r} S_m)^C = H_r$

Therefore $x \epsilon \bigcup_{r=1}^{\infty} H_r$.

Hence the result.