Question

Repeat question 6A, but now finding the transformation directly from the general form of a Möbius map by choosing any three points on $Re(z) + Im(z) = 1$ and their images on $|w| = 1$. Are the mappings of Q6A and Q6B unique? If we demand that the region $Re(z) = Im(z) < 1$ is mapped to the interior of $|w| = 1$, do your maps satisfy this condition? If they do not, find a simple remedy in the form of an additional transformation which is to be carried out.

Answer

$w = \dfrac{\alpha z + \beta}{\gamma z + \delta}$

Without loss of generality we divide through by

$w = \dfrac{az + b}{cz + d}$ for $a,\ b,\ c,\ d$ to be determined.

Pick any three points on $x + y = 1$, say $1,\ \dfrac{1}{2}(1 + i),\ i$.

Now pick any three image points on $|w| = 1$ say $1,\ i1, -1$.

Thus we solve:

$$\left.\begin{array}{ccc} \underline{z} & \longrightarrow & \underline{w} \\ 1 & & 1 \\ \frac{1}{2}(1+i) & \longrightarrow & i \\ i & \longrightarrow & -1 \end{array}\right\} \Rightarrow \begin{array}{rcl} 1 & = & \dfrac{a+b}{1+c} \\ i & = & \dfrac{\frac{1}{2}a(1+i)+b}{\frac{1}{2}(1+i)+c} \\ -1 & = & \dfrac{ai+b}{i+c} \end{array}$$

Solve for $a,\ b,\ c$ to get $\left\{ \begin{array}{l} b = 0,\ a = \dfrac{1}{i} \\ c = -1 - i \end{array} \right.$

Thus $w = \dfrac{z}{iz + (1 - i)}$

This is clearly a different map to Q6A. This is not surprising since we could pick any 3 points and any 3 images for the Möbius map. Thus Möbius maps <u>cannot</u> be unique for this problem: the overall shapes are correct (line $\rightarrow$ circle) but the individual maps of z-points may get mixed up. You probably have a different Möbius map for this reason.

If we require $x + y < 1$ to map to $|w| < 1$ try a test point for

$$w = \frac{\sqrt{2}[(1 + i) - z]}{(1 + i)z} \text{ and } \mathrm{w} = \frac{\mathrm{z}}{\mathrm{iz} + (1 - \mathrm{i})}$$

say $z = 0$,

$$w = \sqrt{2}\left[\frac{(1 + i) - 0}{(1 + i)0}\right] \rightarrow \infty;\ w = 0$$

Thus the solution of Q6A is <u>not</u> sufficient, but the solution of Q6B <u>is</u>.
We can remedy the solution of Q6A by making a further transform $w_4 = \frac{1}{w_3}$. Why? Well $|w| = |w_3| = 1 \to |w_4| = 1$. The boundary is therefore unchanged. But $|w_3| < 1$ is mapped to $|w_4| > 1$ and vice versa. Hence the new mapping $w = \frac{1}{w_3}$ is

$$w = \frac{(1+i)z}{\sqrt{2}[(1+i) - z]}$$

and now the test point $z = 0 \to w = 0$, i.e., to the interior of the unit circle.