Question

Find a Möbius mapping which transforms the points $z = 0, -i, 1$ into $w = i, 1, 0$ respectively. In general how many points and their images do you need to define a Möbius map?

Answer

A Möbius mapping is $w = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ and finite.

Thus we have

$$i = \frac{\alpha \cdot 0 + \beta}{\gamma \cdot + \delta};\ 1 = \frac{-i\alpha + \beta}{-i\gamma + \delta};\ 0 = \frac{-\alpha + \beta}{-\gamma + \delta}$$
$$(1) \qquad\qquad (2) \qquad\qquad (3)$$

(3) $\Rightarrow \beta = \alpha$ (since can't have $\alpha, \beta, \gamma, \delta$ infinite sensibly)
Thus (3) in (1) gives
(1) $\Rightarrow \delta = \dfrac{\beta}{i} = -i\alpha$
Hence
(2) $\Rightarrow -i\gamma + \delta = -i\alpha + \beta = -(1+i)\alpha \Rightarrow \gamma = \dfrac{-(1+i)\alpha - \delta}{-i} = i\alpha$
Thus
$w = \dfrac{\alpha z + \alpha}{i\alpha z - i\alpha} = \dfrac{1}{i}\left(\dfrac{z+1}{z-1}\right) = -i\left(\dfrac{z+1}{z-1}\right)$
Note that α is irrelevant. Therefore in general we require only 3 points and their images for uniqueness.