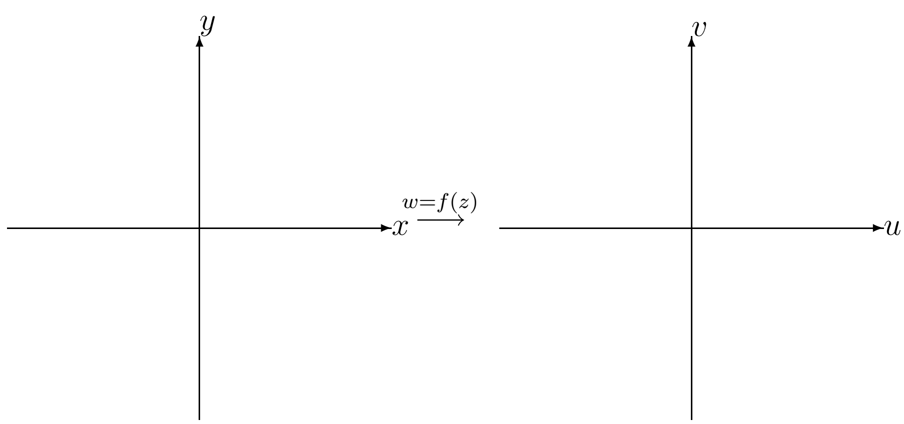
Question
Determine the region of the w plane into which each of the following is mapped by the transformation $w = z^2$.

(i) The first quadrant of the z-plane.

(ii) The region bounded by $x = 1,\ y = 1,\ x + y = 1$ (Hint: calculate the real and imaginary coordinates of the transformed line segments and sketch curves in the w-plane which they parametrise.)

Answer
Let $w = z^2,\ z = re^{i\theta}$. Then $w = r^2 e^{2i\theta}$. Thus points at (r, θ) are rotated by a further angle θ and their modulus stretched by a factor r.

(i) First quadrant of z-plane



All points in 1st quadrant occupy $r > 0,\ 0 \leq \theta \leq \dfrac{\pi}{2}$. Thus all points in $w(= \rho e^{i\phi})$-plane occupy $\rho > 0,\ 0 \leq \phi \leq \pi$, i.e., the upper half of the w-plane.

(ii) Region bounded by $x = 1,\ y = 1m\ x + y = 1.$

If $w = z^2$ we have

$$(u + iv) = (x + iy)(x + iy) = x^2 - y^2 + 2ixy$$

Therefore $\left\{ \begin{array}{rcl} u & = & x^2 - y^2 \\ v & = & 2xy \end{array} \right.$

Thus the lines

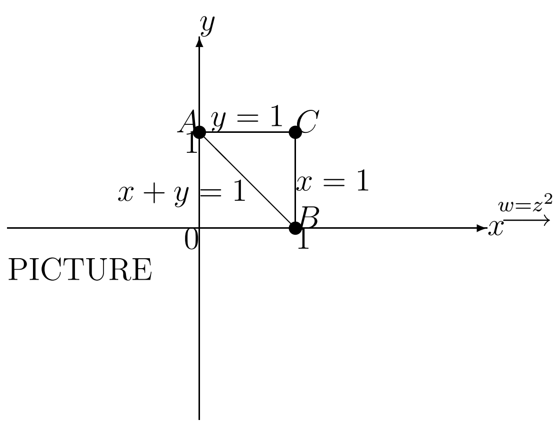
$x = 1 \rightarrow \left\{ \begin{array}{rcl} u & = & 1 - y^2 \\ v & = & 2y \end{array} \right\} \Rightarrow u = 1 - \dfrac{v^2}{4}$

$y = 1 \rightarrow \left\{ \begin{array}{rcl} u & = & x^2 - 1 \\ v & = & 2x \end{array} \right\} \Rightarrow u = \dfrac{v^2}{4} - 1$

$\begin{array}{rcl} x + y & = & 1 \\ \text{or y} & = & 1 - x \end{array} \rightarrow \left\{ \begin{array}{rcl} u & = & x^2 - (1 - x)^2 \\ & = & 2x - 1 \\ v & = & 2x(1 - x) \\ & = & 2x - 2x^2 \end{array} \right\} \Rightarrow v = \dfrac{1}{2}(1 - u^2)$

Thus we have:

(z)



PICTURE

Pick a points inside the region ABC to see where it goes and confirm $ABC \to A'B'C'$ shaded region.

Note that $w' = f(z) = 2z$, so unless $z = 0$ the local angles should be conserved.

Thus is $\angle BAC = \angle ABC = \dfrac{\pi}{4}$

then $\angle B'A'C'$ (or the tangents' angle at A') is $\dfrac{\pi}{4}$

$\angle A'B'C'$ (or the tangents' angle at B') is $\dfrac{\pi}{4}$

Similarly $\angle ACB = \dfrac{\pi}{2}$ and $\angle A'C'B'$ is $\dfrac{\pi}{2}$