Question

Let C be a circle in the z-plane having its centre on the real axis, and suppose further that it passes through $z = 1$ with $z = -1$ as an interior point. Let C be transformed by the mapping $w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Show that this map is not conformal at $z = 1$. Expand $f(z)$ locally in the neighbourhood of $z = 1$ and deduce that the angles between lines which meet at $z = 1$ are doubled. Hence sketch the image of C in the neighbourhood if $w = f(1)$. By picking a few other points and working out their w-images, draw the whole of the transformed C. How does the image of C change if its centre is moved to the upper half plane, but C still passes through $z = 1$?

Answer

(z)
PICTURE

$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \Rightarrow \frac{dw}{dz} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$
so not conformal at $z = \pm 1$ (or 0).
Hence $z = +1$ is a problem point. We expect the angles of C not to be conserved near the image point $w = f(1) = 1$.

We carry out a local analysis:

$$\begin{aligned} f(z) &= f(1) + f'(1)(z-1) + f''(1)\frac{(z-1)^2}{2} + \cdots \\ &= 1 + \frac{f''(1)(z-1)^2}{2} + \cdots \\ \Rightarrow w - 1 &= \frac{(z-1)^2}{2} + \cdots \end{aligned}$$

Thus if we set $(z-1) = re^{i\theta}$
$\Rightarrow w - 1 \approx \frac{r^2}{2}e^{2i\theta}$, i.e., the angles are doubled.
Now what is the angle of C at $z = 1$?
Well it's formed by the tangents of C at 1:
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So, the regular C at $z = 1$ is transformed to a 2π cusp at $w = 1$. All other points of C transform smoothly since 0 and -1 are not on the curve.
By symmetry the new curve is symmetric about the $Re(w)$ axis:
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If the centre of C is removed, but we still pass through $z = 1$, the cusp remains, but is bent to a more aerofoil shape.