Question

Evaluate

(i) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x \, dx$

(ii) $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^7 x \, dx$

(iii) $\int_0^{\frac{\pi}{2}} \sin^x \cos^4 x \, dx$

(iv) $\int_0^1 x^6 91 - x^2)^{\frac{3}{2}} \, dx$

Answer

(i) This is $I_{3,4}$. Use formula of Q6.

$$I_{m,n} = \frac{(m-1)}{(m+n)} I_{m-2,n} \text{ valid for } m, n > 0$$

$$I_{3,4} = \frac{(3-1)}{(3+4)} I_{3-2,4} = \frac{2}{7} I_{1,4}$$

What's $I_{1,4}$?

$$\begin{aligned} I_{1,4} &= \int_0^{\frac{\pi}{2}} \sin x \cos^4 x \, dx \\ &= \left[-\frac{\cos^5 x}{5} \right]_0^{\frac{\pi}{2}} = -0 + \frac{1}{5} = \frac{1}{5} \end{aligned}$$

Therefore $I_{3,4} = \frac{2}{7} \times \frac{1}{5} = \underline{\frac{2}{35}}$.

(ii) This is $I_{5,7}$. Use formula of Q6.

$$I_{5,7} = \frac{(5-1)}{(5+7)} I_{5-2,7} = \frac{4}{12} I_{3,7}$$

$$I_{3,7} = \frac{(3-1)}{(3+7)} I_{3-2,7} = \frac{2}{10} I_{1,7}$$

What's $I_{1,7}$?

$$I_{1,7} = \int_0^{\frac{\pi}{2}} \sin x \cos^7 x \, dx = \left[-\frac{\cos^8 x}{8} \right]_0^{\frac{\pi}{2}} = \frac{1}{8}$$

So

$$I_{3,7} = \frac{2}{10} \times \frac{1}{8}$$

$$I_{5,7} = \frac{4}{12} \times \frac{2}{10} \times \frac{1}{8} = \frac{1}{10 \times 12} = \underline{\frac{1}{120}}$$

(iii) This is $I_{6,4}$. Use formula of Q6.

$$I_{6,4} = \frac{(6-1)}{(6+4)} I_{4,4}$$

$$I_{4,4} = \frac{(4-1)}{(4+4)} I_{2,4}$$

$$I_{2,4} = \frac{(2-1)}{(2+4)} I_{0,4}$$

What's $I_{0,4}$?

$$I_{0,4} = \int_0^{\frac{\pi}{2}} \sin^0 x \cos^4 x \, dx = \int_0^{\frac{\pi}{2}} \cos^4 x \, dx$$

Use result from lecture notes: equations (1.38)

$$\begin{aligned} I_{0,4} &= \left(\frac{4-1}{4}\right) I_{0,2} \\ I_{0,2} &= \left(\frac{2-1}{2}\right) I_{0,0} \\ \Rightarrow \quad &= \left(\frac{2-1}{2}\right) \int_0^{\frac{\pi}{2}} dx \\ &= \frac{\pi}{4} \end{aligned}$$

So $I_{0,4} = \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{3\pi}{16}$

$$I_{2,4} = \frac{1}{6} \times \frac{3\pi}{16}$$

Hence $$I_{4,4} = \frac{3}{8} \times \frac{1}{6} \times \frac{3\pi}{16}$$

$$I_{6,4} = \frac{5}{10} \times \frac{3}{8} \times \frac{1}{6} \times \frac{3\pi}{16} = \underline{\frac{3\pi}{512}}$$

(iv) $\int_0^1 x^6(1-x^2)^{\frac{3}{2}}\,dx$

set
$$\begin{aligned} x &= \sin u \\ \Rightarrow dx &= \cos u\,du \\ \text{and x} = 1 &\rightarrow u = \frac{\pi}{2} \\ x = 0 &\rightarrow u = 0 \end{aligned}$$

So integral becomes

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} du \cos u(\sin u)^6(1-\sin^2 u)^{\frac{3}{2}} \\ =\ &\int_0^{\frac{\pi}{2}} du(\cos u)\sin^6 x(\cos^2 u)^{\frac{3}{2}} \\ =\ &\int_0^{\frac{\pi}{2}} du \cos u \sin^6 x \cos^3 u \\ =\ &\int_0^{\frac{\pi}{2}} du \cos^4 u \sin^6 u = I_{6,4} \end{aligned}$$

We've already done this in part (iii), so the answer is $\underline{\frac{3\pi}{512}}$.