Question Let $I_{m,n} \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$. Prove that $I_{m,n} = I_{n,m}$. By differentiating $\sin^{m-1} x \cos^{n+1} x$, prove that $I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$

Answer $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$

Now $I_{n,m} = \int_0^{\frac{\pi}{2}} \sin^n x \cos^m x \, dx$ (reverse n and m)

Now $\left.\begin{array}{c} \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha \\ \sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha \end{array}\right\}$ standard trig. results

So in $I_{n,m}$ substitute $x = \frac{\pi}{2} - u$.

$$\Rightarrow \left\{ \begin{array}{ccc} dx & = & -du \\ x = \frac{\pi}{2} & \to & u = 0 \\ x = 0 & \to & u = \frac{\pi}{2} \end{array} \right.$$

Then

$I_{n,m} = -\int_{\frac{\pi}{2}}^0 du \sin^n\left(\frac{\pi}{2} - u\right) \cos^m\left(\frac{\pi}{2} - u\right)$

$\Rightarrow I_{n,m} = -\int_{\frac{\pi}{2}}^0 du \cos^n u \sin^m u$

$\Rightarrow I_{n,m} = +\int_0^{\frac{\pi}{2}} du \sin^m u \cos^n u$ (reverse sign by integral property $\int_a^b = -\int_b^a$)

$\Rightarrow \underline{I_{n,m} = I_{m,n}}$ (as defined above) as required.

Differentiate $sin^{m-1} x \cos^{n+1} x$

$$\begin{aligned} \frac{d}{dx}(\sin^{m-1} x \cos^{n+1} x) &= (m-1)\sin^{m-2} x \cos x \cos^{n+1} x \\ &- (n+1)\cos^n x \sin x \sin^{m-1} x \\ &= (m-1)\sin^{m-2} x \cos^{n+1} x \\ &- (n+1)\underbrace{\sin^m x \cos^n x}_{\text{look familiar?}} \end{aligned}$$

In other words, rearranging for $\sin^m x \cos^n x$ we have:

$(n+1)\sin^m x \cos^n x = (m-1)\sin^{m-2} x \cos^{n+2} x - \frac{d}{dx}(\sin^{m-1} x \cos^{n+1} x)$

or, integrating both sides:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx &= \frac{(m-1)}{(n+1)} \int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^{n+2} x \, dx \\ &- \frac{1}{n+1} \int_0^{\frac{\pi}{2}} \frac{d}{dx} (\sin^{m-1} x \cos^{n+1} x) \, dx\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx &= \frac{(m-1)}{(n+1)} \int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^{n+2} x \, dx \\ &- \frac{1}{n+1} - \sin^{m-2} x \cos^{n+1} x]_0^{\frac{\pi}{2}}\end{aligned}$$

(since $\int_a^b dx(\frac{df}{dx}) = [f]_a^b$)

i.e. $I_{m,n} = \frac{(m-1)}{(n+1)} \int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^{n+2} x \, dx - 0$

Now write

$$\begin{aligned}\sin^{m-2} x \cos^{n+2} x &= \sin^{m-2} x \cos^n x \cos^2 x \\ &= \sin^{m-2} x \cos^n x (1 - \sin^2 x) \\ &= \sin^{m-2} x \cos^n x - \sin^m x \cos^n x\end{aligned}$$

Thus, we have

$$\begin{aligned}I_{m,n} &= \frac{(m-1)}{(n+1)} \int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^n x \, dx \\ &- \frac{(m-1)}{(n+1)} \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx \\ \Rightarrow I_{m,n} &= \frac{(m-1)}{(n+1)} I_{m-2,n} - \frac{(m-1)}{(n+1)} I_{m,n} \\ &\Rightarrow \underline{I_{m,n} = \frac{(m-1)}{(m+n)} I_{m-2,n}}\end{aligned}$$