Question
Find the rms value in the interval $0 \le x \le 2\pi$ of the function

$$e^{-\frac{x}{5}} \sin\left(\frac{x}{2}\right).$$

Answer

$$\begin{aligned} \text{rms} &= \sqrt{\frac{1}{2\pi - 0}\int_0^{2\pi} \left(e^{-\frac{x}{5}} \sin \frac{x}{2}\right)^2 dx} \\ &= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{2x}{5}} \sin^2 \frac{x}{2}\, dx} \end{aligned}$$

Consider $K = \int_0^{2\pi} e^{-\frac{2x}{5}} \sin^2 \frac{x}{2}\, dx$

The first thing to do is get rid of the $\sin^2 \frac{x}{2}$ term by using the double angle formula.

$$\begin{aligned} 1 - 2\sin^2\left(\frac{x}{2}\right) &= \cos x \\ \Rightarrow \sin^2\left(\frac{x}{2}\right) &= \frac{1}{2}(1 - \cos x) \end{aligned}$$

So

$$\begin{aligned} K &= \frac{1}{2}\int_0^{2\pi} e^{-\frac{2x}{5}} (1 - \cos x)\, dx \\ &= \frac{1}{2}\int_0^{2\pi} e^{-\frac{2x}{5}}\, dx - \frac{1}{2}\int_0^{2\pi} e^{-\frac{2x}{5}} \cos x\, dx \\ &= \quad \text{Easy} \qquad\qquad \text{Hard} \\ &= \frac{1}{2}\left[\frac{-5}{2} e^{-\frac{2x}{5}}\right]_0^{2\pi} - \frac{1}{2} I \quad \text{say} \\ &= -\frac{5}{4}(e^{\frac{-4\pi}{5}} - 1) - \frac{1}{2} I \end{aligned}$$

Consider I now:

$$I = \int_0^{2\pi} e^{-\frac{2x}{5}} \cos x\, dx$$

This is difficult to do, so integrate by parts.

$$\begin{aligned} u &= e^{-\frac{2x}{5}} & \frac{dv}{dx} &= \cos x \\ \frac{du}{dx} &= -\frac{2}{5} e^{-\frac{2x}{5}} & v &= \sin x \end{aligned}$$

Then

$$I = [e^{-\frac{2x}{5}} \sin x]_0^{2\pi} - \int_0^{2\pi} \left(-\frac{2}{5}e^{-\frac{2x}{5}}\right) \sin x \, dx$$

So $I = 0 + \frac{2}{5}\int_0^{2\pi} e^{-\frac{2x}{5}} \sin x \, dx$
$\Rightarrow I = \frac{2}{5}J$
Where

$$J = \int_0^{2\pi} e^{-\frac{2x}{5}} \sin x \, dx$$

This is still hard, but we can integrate by parts again. Why? I can turn the sin back into a cos $\Rightarrow$ I'm back with I again and if I've done it correctly, I have a reduction formula:

$$\begin{array}{cc} u = e^{-\frac{2x}{5}} & \frac{dv}{dx} = \sin x \\ \frac{du}{dx} = -\frac{2}{5}e^{-\frac{2x}{5}} & v = -\cos x \end{array}$$

So

$$\begin{aligned} J &= [-e^{-\frac{2x}{5}} \cos x]_0^{2\pi} - \int_0^{2\pi} \frac{2}{5} \cos x e^{-\frac{2x}{5}} \, dx \\ &= -3^{-\frac{4\pi}{5}} \cos 2\pi + e^0 \cos 0 - \frac{2}{5}\int_0^{2\pi} \cos x e^{-\frac{2x}{5}} \, dx \\ &= -e^{-\frac{4\pi}{5}} + 1 - \frac{2}{5}I \end{aligned}$$

Thus

$$\begin{aligned} I &= \frac{2}{5}J = \frac{2}{5}\left[1 - e^{-\frac{4\pi}{5}} - \frac{2}{5}I\right] \\ \Rightarrow \quad & I = \frac{2}{5}(1 - e^{-\frac{4\pi}{5}}) - \frac{4}{25}I \\ \Rightarrow \quad & \frac{29}{5}I = \frac{2}{5}(1 - e^{-\frac{4\pi}{5}}) \\ \Rightarrow \quad & I = \frac{2 \times 25}{29 \times 5}(1 - e^{-\frac{4\pi}{5}}) \\ & = \frac{10}{29}(1 - e^{-\frac{4\pi}{5}}) \end{aligned}$$

Now look back to the original integral K:

$$\begin{aligned}
K &= -\frac{5}{4}(e^{-\frac{4\pi}{5}} - 1) - \frac{1}{2}I \\
&= \frac{5}{4}(1 - e^{-\frac{4\pi}{5}}) - \frac{1}{2}\frac{10}{29}(1 - e^{-\frac{4\pi}{5}}) \\
&= (1 - e^{-\frac{4\pi}{5}})\left(\frac{5}{4} - \frac{20}{58}\right) \\
&= (1 - e^{-\frac{4\pi}{5}})\left(\frac{58 \times 5 - 80}{4 \times 58}\right) \\
&= (1 - e^{-\frac{4\pi}{5}})\left(\frac{210}{232}\right) \\
&= (1 - e^{-\frac{4\pi}{5}})\left(\frac{105}{116}\right)
\end{aligned}$$

Now RMS value

$$= \sqrt{\frac{1}{2\pi} \times K}$$

$$= \underline{\sqrt{\frac{105}{232\pi}(1 - e^{-\frac{4\pi}{5}})}}$$