Question

Find a reduction formula for $I_N = \int (\ln x)^n\,dx$.

Evaluate $\int_1^e (\ln x)^4\,dx$.

Answer $I_n = \int (\ln x)^n\,dx$

Integrate by parts with

$$\begin{array}{cc} u = (\ln x)^n & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = n(\ln x)^{n-1} \times \dfrac{1}{x} & v = x \end{array}$$

Therefore
$$\begin{aligned} I_n &= x(\ln x)^n - \int dx \frac{n(\ln x)^{n-1}}{x} \times x \\ &= x(\ln x)^n - n\int dx (\ln x)^{n-1} = x(\ln x)^n - nI_{n-1} \end{aligned}$$

Hence $\underline{I_n = x(\ln x)^n - nI_{n-1}}$ is the reduction formula.

If $I_4 = \int_1^e (\ln x)^4\,dx$ then inserting the integration limits into the above expression we have, with $n = 4$.

$$\begin{aligned} I_4 &= [x(\ln x)^4]_1^e - 4I_3 \\ I_3 &= [x(\ln x)^3]_1^e - 3I - 2 \\ I - 2 &= [x(\ln x)^2]_1^e - 2I_1 \\ I_1 &= [x(\ln x)]_1^e - I_0 \end{aligned}$$

What's I_0?

$I - 0 = \int_1^e (\ln x)^0\,dx = \int_1^e\,dx = \underline{e - 1}$

Therefore

$$\begin{aligned} I_1 &= (e\ln e - 1\ln 1) - (e - 1) \quad (\ln 1 = 0,\ \ln e = 1) \\ &= e - (e - 1) = \underline{1} \\ \text{Hence } \mathrm{I}_2 &= [e(\ln e)^2 - 1(\ln 1)^2] - 2I_1 \\ &= e - 2 \times 1 = \underline{e - 2} \\ \text{Hence } \mathrm{I}_3 &= [e(\ln e)^3 - 1(\ln 1)^3] - 3I_2 \\ &= e - 3 \times (e - 2) = \underline{6 - 2e} \\ \text{Hence } \mathrm{I}_4 &= [e(\ln e)^4 - 1(\ln 1)^4] - 4I_3 \\ &= e - 4 \times (6 - 2e) = \underline{9e - 24} \end{aligned}$$