Question Find a reduction formula for $I_N = \int_0^\infty \frac{dx}{(1+x^2)^n}$. Work out the value of I_1.
Hence evaluate this integral for arbitrary positive integer n,

Answer $I_n = \int_0^\infty \frac{dx}{(x^2+1)^n}$
(Assume it converges for n positive and > 1)
Integrate by parts with

$$u = \frac{1}{(1+x^2)^n} \qquad \frac{dv}{dx} = 1$$
$$\frac{du}{dx} = \frac{-n \times 2x}{(1+x^2)^{n+1}} \qquad v = x$$

Therefore
$$\begin{aligned} I_n &= \left[\frac{x}{(1+x^2)^n}\right]_0^\infty - \int_0^\infty dx x \cdot \frac{(-2nx)}{(1+x^2)^{n+1}} \\ &= 0 (if n > 1) + 2n \int_0^\infty \frac{dx x^2}{(1+x^2)^{n+1}} \end{aligned}$$

So $I_n = 2n \int_0^\infty \frac{dx x^2}{(1+x^2)^{n+1}}$
What now? Use a similar trick to Q3.

$$x^2 = (1+x^2) - 1$$

So
$$\begin{aligned} I_n &= 2n \int_0^\infty \frac{dx[(1+x^2)-1]}{(1+x^2)^{n+1}} \\ &= 2n \int_0^\infty \frac{dx(1+x^2)}{(1+x^2)^{n+1}} - 2n \int_0^\infty \frac{dx}{(1+x^2)^{n+1}} \\ &= 2n \underbrace{\int_0^\infty \frac{dx}{(1+x^2)^n}}_{I_n} - 2n \underbrace{\int_0^\infty \frac{dx}{(1+x^2)^{n+1}}}_{I_{n+1}} \end{aligned}$$

Therefore $I_n = 2nI_n - 2nI_{n+1}$
or $I_{n+1} = \frac{(2n-1)}{2n} I_n$
This can be rewritten by putting $n+1 \to n$

or
$$I_n = \frac{(2(n-1)-1)}{2(n-1)} I_{n-1}$$
$$\underline{I_n = \left(\frac{2n-3}{2n-2}\right) I_{n-1}} \quad (\star)$$

What's I_1?

$$I_1 = \int_0^\infty \frac{dx}{(1+x^2)^1} = \int_0^\infty \frac{dx}{(1+x^2)} = [\arctan x]_0^\infty + \frac{\pi}{2}$$

Hence applying $(\star)$ recursively:

$I_n = \left(\frac{2n-3}{2n-2}\right) I_{n-1}$

$I_{n-1} = \left(\frac{2n-5}{2n-4}\right) I_{n-2}$

$I_{n-2} = \left(\frac{2n-7}{2n-6}\right) I_{n-3}$

$I_{n-3} =$ ....*etc.*

$\vdots$

$\vdots$

$I_2 = \left(\frac{2\times 2-3}{2\times 2-2}\right) I_1 = \frac{1}{2}\frac{\pi}{2}$

Recursion. This assumes that n is a <u>positive integer</u>.

Hence

$$\underline{I_n = \frac{(2n-3)}{(2n-2)} \times \frac{(2n-5)}{(2n-4)} \times \frac{(2n-7)}{(2n-6)} \times \cdots \times \frac{1}{2} \times \frac{\pi}{2}}$$