Question if $I_n = \int (x^2+3)^n \, dx$, show that $(2n+1)I_n = x(x^2+3)^n + 6nI_{n-1}$.

Answer $I_n = \int (x^2+3)^n \, dx$
We want to reduce the powers of n. So, integrate by parts with

$$\begin{array}{cc} u = (x^2+3)^n & \dfrac{dv}{dx} = 1 \\ \dfrac{du}{dx} = n(x^2+3)^{n-1}2x & v = x \end{array}$$

$$\begin{aligned} I_n &= x(x^2+3) - \int dx \cdot x \cdot n \cdot 2x \cdot (x^2+3)^{n-1} \\ &= x(x^2+3) - 2n \int dx x^2 (x^2+3)^{n-1} \end{aligned}$$

Now $x^2 = (x^2+3) - 3$. This is the key step
So

$$\begin{aligned} I_n &= x(x^2+3) - 2n \int dx[(x^2+3) - 3](x^2+30^{n-1} \\ &= x(x^2+3) - 2n \underbrace{\int dx (x^2+3)^n}_{I_n} + 6n \underbrace{\int dx (x^2+3)^{n-1}}_{I_{n-1}} \end{aligned}$$

So $I_n = x(x^2+3) - 2nI_n + 6nI_{n-1}$
Rearrange for I_n
$\underline{(2n+1)I_n = x(x^2+3) + 6nI_{n-1}}$ as required.