Question

Find a reduction formula for $I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta$ and evaluate I_6 and I_7.

Answer

$I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta \quad (\star) = \int_0^{\frac{\pi}{2}} \sin^{n-1} \theta \sin \theta \, d\theta$

Integrate by parts

$$u = \sin^{n-1} \theta, \; \frac{dv}{d\theta} = \sin \theta$$

$\Rightarrow \frac{du}{d\theta} = (n-1)\sin^{n-2} \theta \cos \theta, \; v = -\cos \theta.$

Therefore

$$\begin{aligned} I_n &= [-\sin^{n-1} \theta \cos \theta]_0^{\frac{\pi}{2}} - (1-n) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \cos^2 \theta d\theta \\ &= 0 + (n-1) \int -0^{\frac{\pi}{2}} \sin^{n-2} \theta (1 - \sin^2 \theta) d\theta \\ & \quad \text{since } 1 = \sin^2 \theta + \cos^2 \theta \\ I_n &= (n-1) \int_0^{\frac{\pi}{2}} sin^{n-2} \theta \, d\theta - (n-1) \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta \\ \Rightarrow I_n &= (n-1) I_{n-1} - (n-1) I_n \end{aligned}$$

by comparison with $(\star)$

So solve for I_n

$$nI_n = (n-1)I_{n-2} \Rightarrow I_n = \frac{9n-1)}{n} I_{n-2}$$

Works for $(n > 1)$. Reduction formula.

$I_6 = \frac{5}{6} I_4; \; I_4 = \frac{3}{4} I_2; \; I_2 = \frac{1}{2} I_0$

What's I_0?

$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \, dx = \int_0^{\frac{\pi}{2}} \, dx = \frac{\pi}{2}$

Thus $I_2 = \frac{\pi}{4} \Rightarrow I_4 = \frac{3}{4} \times \frac{\pi}{4} \Rightarrow I_4 = \frac{3}{4} \times \frac{\pi}{4} \Rightarrow I_6 = \frac{5}{6} \times \frac{3}{4} \times \frac{\pi}{4} = \underline{\frac{5\pi}{32}}$

$I_7 = \frac{6}{7} I_5; \; I_5 = \frac{4}{5} I_3; \; I_3 = \frac{2}{3} I_1$

What's I_1?

$I_1 = \int_0^{\frac{\pi}{2}} \sin \theta \, d\theta = [-\cos \theta]_0^{\frac{\pi}{2}} = 1$

Thus $I_3 = \frac{2}{3} \times 1, \; I_5 = \frac{4}{5} \times \frac{2}{3} \times 1, \; I_7 = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \underline{\frac{16}{35}}$