Question

Show from the definition of *open set* in $\mathbf{R}^n$ that if A and B are open sets then $A \cap B$ and $A \cup B$ are open sets. Show the same results hold with *open* replaced by *closed*. Deduce that the intersection and union of a finite collection of open (closed) sets is open (closed). Show from the definition that the *union* of an infinite collection of open sets is necessarily open. By considering the open intervals $(-\frac{1}{n}, \frac{1}{n})$ in $\mathbf{R}$ or otherwise, show that the *intersection* of an infinite collection of open sets need not be open. Show that the *intersection* of an infinite collection of closed sets is necessarily closed, and give an example of an infinite collection of closed sets whose *union* is not closed.

Answer

A, B open: Let $p \in A \cap B$.
Since A, B are open there exist α, $\beta > 0$ such that $B_\alpha(p) \subset A$, $B_\beta(p) \in B$.
Let $\epsilon = \min\{\alpha, \beta\}$: Then $B_\epsilon(p) \subset A \cap B$. Thus $A \cap B$ open.
Let $q \in A \cup B$. If $q \in A$ there exists $\alpha > 0$ such that $B_\alpha(q) \subset A \subseteq A \cup B$. If $q \in B$ there exists $\beta > 0$ such that $B_\beta(q) \subset B \subseteq A \cup B$. Thus $A \cup B$ open.

$$\begin{aligned} A, B\text{closed} &\Leftrightarrow \Re^n \backslash A, \Re^n \backslash B \text{ open (by definition)} \\ &\Rightarrow (\Re^n \backslash A) \cap (\Re^n \backslash B) \text{ open by the above} \\ &\Rightarrow \Re^n \backslash (A \cup B) \text{ open} \\ &\Rightarrow A \cap B \text{ closed (by definition)}. \end{aligned}$$

Likewise for $A \cap B$, using $\Re^n \backslash (A \cap B) = (\Re^n \backslash A) \cup (\Re^n \backslash B)$.
Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a collection of open sets, and suppose $p \in \bigcup_{\lambda \in \Lambda} U_\lambda = W$, say.
This means there is at least one $\lambda \in Lambda$ with $p \in U_\lambda$. Since U_λ is open there exists $\epsilon > 0$ with $B_\epsilon(p) \subset U_\lambda \subseteq W$. Hence W is open.
If $I_n = (-\frac{1}{n}, \frac{1}{n})$ then $\bigcap_n = 1^\infty I_n$ consists of $\{0\}$ only: this is not open in $\Re$.
If $\{C_\lambda\}_{\lambda \in \Lambda}$ is a collection of closed sets, then $\{\Re^n \backslash C_\lambda\}_{\lambda \in \Lambda}$ is a collection of open sets, with $\Re^n \backslash \bigcap_{\lambda \in \Lambda} C_\lambda = U_{\lambda \in \Lambda}(\Re^n - C_\lambda)$; then use above result.
If $J_n = \Re \backslash I_n = (-\infty, -frac1n] \cup [\frac{1}{n}, \infty)$ which is closed in $\Re$ then $\bigcup_{n=1}^\infty = \Re^n \backslash \{0\}$ which is not closed in $\Re$.