Question

Let $[a, b]$ be a closed interval in R, and let A be a given subset of R such that $a \in A$ and $b \notin A$. The purpose of this exercise is to show that A cannot be both open and closed (so the only sets in R which are both open and closed are the empty set $\emptyset$ and the real line R itself). Let $V = A \cap [a, b]$, and let ξ be the *supremum* (least upper bound) of V. Clearly $a < \xi < b$ since $a \in V$ but $b \notin V$. Does ξ belong to A or not ? Show that if A is open then ξ does not belong to A, and if A is closed then ξ does not belong to the complement of A. Therefore A cannot be both open and closed.

Answer

Suppose $\xi \in A$. Since A is open, there exists $\epsilon > 0$ with $(\xi - \epsilon, \xi + \epsilon) \subset A$ and with $\xi + \epsilon < b$, so $\xi + \epsilon \in V$ and of course $\xi + \epsilon > \xi$.
This contradicts ξ being an upper bound for V.
Let $B = \Re \backslash A$. Suppose $\xi \in B$, since B is open , there exists $\epsilon > 0$ with $(\xi - \epsilon, \xi + \epsilon) \subset B$ and with $\xi - \epsilon > a$, so $\xi - \epsilon$ is an upper bound for V but $\xi - \epsilon < \xi$.
This contradicts ξ being the least upper bound for V. So ξ can belong to neither A nor $B = \Re \backslash A$; impossible, so A cannot be both open and closed.